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Subject – Vector Calculus

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Chapter 2: Integrals

Topic- Line integral

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Integrals

* Line Integral -

Let f be a function defined on a curve c of finite length. Then the line integral of f along c is defined as

① $$\int_c f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i,y_i)\,\Delta s_i$$

[For two dimensions that is in the plane]

② $$\int_c f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i,y_i,z_i)\,\Delta s_i$$

[For three dimensions]

* How to evaluate a line integral -

To integrate a continuous function $f(x,y,z)$ over a curve c

i) Find a smooth parametrization of c,

$$\bar{r}(t) = g(t)\,\bar{i} + h(t)\,\bar{j} + k(t)\,\bar{k}, \quad a<t<b$$

ii) Evaluate the integral as

$$\int_c f(x,y,z)\,ds = \int_a^b f(g(t), h(t), k(t)) \cdot |\bar{v}(t)|\,dt$$

Remark - Additive property -

If a piecewise smooth curve c is made by joining a finite number of smooth curves $c_1, c_2, \ldots, c_n$ end to end, then the integral of a function over c is the sum of the integrals over the curves that make it up.

$$\int_c f\,ds = \int_{c_1} f\,ds + \int_{c_2} f\,ds + \cdots + \int_{c_n} f\,ds$$

* Parametric eqns of some standard curves -

i) For ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ we the parametric eqn as

i) For counter-clockwise $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$

ii) For clockwise $x = a\cos t$, $y = -b\sin t$, $0 \le t \le 2\pi$

2] For circle $x^2+y^2=a^2$ use the parametric eqn

(i) For counter-clockwise $x=a\cos t,\ y=a\sin t$

$0\le t\le 2\pi$

(ii) For clockwise $x=a\cos t,\ y=-a\sin t,\ 0\le t\le 2\pi$

3] $y=f(x)$ use the parametric eqn as

$x=t,\ y=f(t)$

4] $x=g(y)$ use $x=g(t),\ y=t$

5] Line segment from (x_0,y_0,z_0) to (x_1,y_1,z_1) use the parametric eqn as

$$\bar{r}(t)=(1-t)(x_0,y_0,z_0)+t(x_1,y_1,z_1)$$

$$0\le t\le 1$$

or

$$x=(1-t)x_0+tx_1,\ y=(1-t)y_0+ty_1,\ z=(1-t)z_0+tz_1$$

$$0\le t\le 1$$

Ex. Integrate $f(x,y,z)=x-3y^2+z$ over the line segment c joining the origin to the point $(1,1,1)$.

$\Rightarrow$ $\bar{r}(t)=(1-t)(0,0,0)+t(1,1,1),\quad 0\le t\le 1$

$$=(t,t,t)$$

$$\bar{r}(t)=t\bar{i}+t\bar{j}+t\bar{k}=x\bar{i}+y\bar{j}+z\bar{k}$$

$$\bar{v}(t)=\bar{i}+\bar{j}+\bar{k}$$

$$|\bar{v}(t)|=\sqrt{3}$$

$x=t,\ y=t,\ z=t$

at $(0,0,0)\Rightarrow t=0$ & at $(1,1,1)\Rightarrow t=1$

$$\int_c f(x,y,z)\,ds=\int_{t=0}^{1} f(t,t,t)\,|\bar{v}(t)|\,dt$$

$$=\int_0^1 (t-3t^2+t)\sqrt{3}\,dt$$

$$=\sqrt{3}\left[\frac{t^2}{2}-t^3+\frac{t^2}{2}\right]_0^1$$

$$=\sqrt{3}\left[\left(\frac{1}{2}-1+\frac{1}{2}\right)-0\right]=0$$

Ex. Find the line integral of $\int (x+y+z)\,ds$ over the straight line segment from $(1,2,3)$ to $(0,-1,1)$.

$\Rightarrow$ $\bar{r}(t) = (1-t)(1,2,3) + t(0,-1,1)$, $0 \le t \le 1$

$= (1-t,\ 2-2t,\ 3-3t) + (0,-t,t)$

$= (1-t,\ 2-3t,\ 3-2t)$

$\bar{r}(t) = (1-t)\bar{i} + (2-3t)\bar{j} + (3-2t)\bar{k}$

$\bar{v}(t) = -\bar{i} - 3\bar{j} - 2\bar{k}$

$|\bar{v}(t)| = \sqrt{1+9+4} = \sqrt{14}$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$x = 1-t,\ y = 2-3t,\ z = 3-2t$

for $(1,2,3) \Rightarrow x = 1-t \Rightarrow 1 = 1-t \Rightarrow t=0$

for $(0,-1,1) \Rightarrow x = 1-t \Rightarrow 0 = 1-t \Rightarrow t=1$

$$\int_C (x+y+z)\,ds = \int_{t=0}^{1} [(1-t)+(2-3t)+(3-2t)]\,|\bar{v}(t)|\,dt$$

$$= \int_0^1 (6-6t)\sqrt{14}\,dt$$

$$= \sqrt{14}\left[6t - 6\frac{t^2}{2}\right]_0^1$$

$$= \sqrt{14}\,[(6-3)-0]$$

$$= 3\sqrt{14}$$

Ex. Evaluate $\int_C (xy-4z)\,ds$ where C is the line segment from $(1,1,0)$ to $(2,3,-2)$

$\Rightarrow$ $\bar{r}(t) = (1-t)(1,1,0) + t(2,3,-2)$, $0 \le t \le 1$

$= (1-t,\ 1-t,\ 0) + (2t,\ 3t,\ -2t)$

$= (1-t+2t,\ 1-t+3t,\ 0-2t)$

$= (1+t,\ 1+2t,\ -2t)$

$\bar{r}(t) = (1+t)\bar{i} + (1+2t)\bar{j} - 2t\,\bar{k}$

$\bar{v}(t) = \frac{d\bar{r}}{dt} = \bar{i} + 2\bar{j} - 2\bar{k}$ $\qquad \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$|\bar{v}(t)| = \sqrt{1+4+4} = 3$

$x = 1+t, \quad y = 1+2t, \quad z = -2t$

at $(1,1,0) \Rightarrow x = 1+t \Rightarrow 1 = 1+t \Rightarrow t = 0$

at $(2,3,-2) \Rightarrow 2 = 1+t \Rightarrow t = 1$

$$\int_C (xy - 4z)\,ds = \int_0^t [(1+t)(1+2t) - 4(-2t)]\,|\bar{v}(t)|\,dt$$

$$= \int_0^1 [1 + 2t + t + 2t^2 + 8t]\,3\,dt$$

$$= 3\int_0^1 [2t^2 + 11t + 1]\,dt$$

$$= 3\left[2\frac{t^3}{3} + 11\frac{t^2}{2} + t\right]_0^1$$

$$= 3\left[\frac{2}{3} + \frac{11}{2} + 1\right] - 0$$

$$= 3\left[\frac{37}{6} + 1\right]$$

$$= 3\left[\frac{43}{6}\right]$$

$$= \frac{43}{2}$$

Ex. Evaluate $\int_C (3x^2 - 2y)\,ds$ where c is the line segment from $(3,6)$ to $(1,-1)$

$\Rightarrow \bar{r}(t) = (1-t)(3,6) + t(1,-1), \qquad 0 \le t \le 1$

$= (3-3t, 6-6t) + (t, -t)$

$= (3-2t, 6-7t)$

$\bar{r}(t) = (3-2t)\bar{i} + (6-7t)\bar{j}$

$\bar{v}(t) = -2\bar{i} - 7\bar{j}$ $\qquad \bar{r} = x\bar{i} + y\bar{j}$

$|\bar{v}(t)| = \sqrt{4+49} = \sqrt{53}$

$x = 3-2t \quad , \quad y = 6-7t$

$$\int_C (3x^2 - 2y)\, ds = \int_0^1 [3(3-2t)^2 - 2(6-7t)]\, |\bar{v}(t)|\, dt$$

$$9-12t+4t^2$$

$$= \int_0^1 [27 - 36t + 12t^2 - 12 + 14t] \sqrt{53}\, dt$$

$$= \int_0^1 [12t^2 - 22t + 15] \sqrt{53}\, dt$$

$$= \sqrt{53} \left[12\frac{t^3}{3} - 22\frac{t^2}{2} + 15t \right]_0^1$$

$$= \sqrt{53} [4 - 11 + 15 - 0]$$

$$= \sqrt{53} [8]$$

$$= 8\sqrt{53}$$

Ex. Integrate $f(x,y,z) = x - 3y^2 + z$ over C, where C is the curve joining the origin to the point $(1,1,0)$ and to $(1,1,1)$

$\Rightarrow$ 1) Take C_1 is from origin $(0,0,0)$ to $(1,1,0)$

$$\bar{r}(t) = (1-t)(0,0,0) + t(1,1,0) \quad , 0 \le t \le 1$$

$$= (t, t, 0)$$

$$\bar{r}(t) = t\bar{i} + t\bar{j} + 0\bar{k}$$

$$\bar{v}(t) = \bar{i} + \bar{j} + 0\bar{k}$$

$$|\bar{v}(t)| = \sqrt{1+1} = \sqrt{2}$$

$$x = t, \ y = t, \ z = 0$$

$$\int_{C_1} f(x,y,z)\, ds = \int_{C_1} (x - 3y^2 + z)\, ds$$

$$= \int_0^1 (t - 3t^2)\, |\bar{v}(t)|\, dt$$

$$= \int_0^1 (t - 3t^2) \sqrt{2}\, dt$$

$$= \left[\frac{t^2}{2} - 3\frac{t^3}{3}\right]_0^1 \sqrt{2}$$

$$= \sqrt{2}\left[\frac{1}{2} - 1\right]$$

$$I_1 = \sqrt{2}\left(-\frac{1}{2}\right) = -\frac{1}{\sqrt{2}} \quad \text{--- (1)}$$

(2) C_2 is the line joining the point $(1,1,0)$ to $(1,1,1)$

$$\bar{r}(t) = (1-t)(1,1,0) + t(1,1,1)$$

$$= (1-t, 1-t, 0) + (t, t, t)$$

$$= (1, 1, t)$$

$$\bar{r}(t) = \bar{i} + \bar{j} + t\bar{k}$$

$$\bar{v}(t) = \bar{k}$$

$$|\bar{v}| = \sqrt{0+0+1} = 1$$

$x = 1, y = 1, z = t$

$$\int_{C_2} f(x,y,z)\,ds = \int_C (x - 3y^2 + z)\,ds$$

$$= \int_0^1 (1 - 3 + t)\,|\bar{v}(t)|\,dt$$

$$= \int_0^1 (-2+t)\cdot 1\,dt$$

$$I_2 = \left[-2t + \frac{t^2}{2}\right]_0^1$$

$$= \left(-2 + \frac{1}{2}\right) - 0$$

$$= -\frac{3}{2}$$

Here $C = C_1 \cup C_2$

$$\therefore \int_C f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds$$

$$= -\frac{1}{\sqrt{2}} - \frac{3}{2}$$

Ex. Integrate $f(x,y,z) = x + \sqrt{y} - z^2$ over the path from $(0,0,0)$ to $(1,1,1)$ given by

$C_1: \bar{r}(t) = t\bar{k}, \quad 0 \le t \le 1$

$c_2: \bar{r}(t) = t\bar{j} + \bar{k} \quad , 0 \le t \le 1$

$c_3: \bar{r}(t) = t\bar{i} + \bar{j} + \bar{k} \quad , 0 \le t \le 1$

$\Rightarrow$ Here $c = c_1 \cup c_2 \, c_3$

$\therefore \int_c f \, ds = \int_{c_1} f + \int_{c_2} f + \int_{c_3} f$

$c_1: \bar{r}(t) = t\bar{k} = 0\bar{i} + 0\bar{j} + t\bar{k}$

$\bar{v}(t) = \bar{k} \Rightarrow |\bar{v}| = 1$

$x = 0, y = 0, z = t$

$c_2: \bar{r}(t) = t\bar{j} + \bar{k}$

$\bar{v}(t) = 1\bar{j} + 0\bar{k} \Rightarrow |\bar{v}| = 1$

$x = 0, y = t, z = 1$

$c_3: \bar{r}(t) = t\bar{i} + \bar{j} + \bar{k}$

$\bar{v}(t) = 1\bar{i} + 0 + 0 \Rightarrow |\bar{v}| = 1$

$x = t, y = 1, z = 1$

$$\int_c f(x,y,z)\,ds = \int_c (x + \sqrt{y} - z^2)\,ds$$

$$= \int_{c_1} (x + \sqrt{y} - z^2)\,ds + \int_{c_2} (x + \sqrt{y} - z^2)\,ds + \int_{c_3} (x + \sqrt{y} - z^2)\,ds$$

$$= \int_0^1 (0 + 0 - t^2)\,|\bar{v}|\,dt + \int_0^1 (0 + \sqrt{t} - 1)\,|\bar{v}(t)|\,dt$$

$$+ \int_0^1 (t + 1 - 1)\,|\bar{v}(t)|\,dt$$

$x^n = \frac{x^{n+1}}{n+1}$

$\sqrt{t} = t^{\frac{1}{2}} = \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{t^{3/2}}{3/2}$

$$= \int_0^1 -t^2 \cdot 1\,dt + \int_0^1 (\sqrt{t} - 1) \cdot 1\,dt + \int_0^1 t\,(1)\,dt$$

$$= -\left[\frac{t^3}{3}\right]_0^1 + \left[\frac{t^{3/2}}{3/2} - t\right]_0^1 + \left[\frac{t^2}{2}\right]_0^1$$

$$= -\frac{1}{3} + \frac{2}{3}(1) - 1 + \frac{1}{2}$$

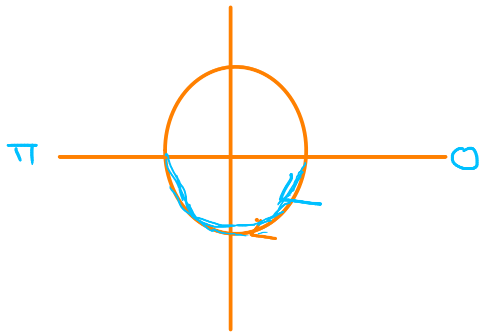
$$= \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$$

Ex. Evaluate $\int_C (2yx^2 - 4x)\, ds$ where c is the lower half of the circle centered at the origin of radius 3 with clockwise direction

$\Rightarrow$ parametric eqn of circle in clockwise direction is $x^2+y^2=a^2$

$x = a\cos t \;,\; y = -a\sin t$

here radius $= a = 3$



$x = 3\cos t \,,\; y = -3\sin t$

$0 \le t \le \pi$

$\bar{r}(t) = x\bar{i} + y\bar{j}$

$\bar{r}(t) = 3\cos t\,\bar{i} - 3\sin t\,\bar{j}$

$\bar{v}(t) = -3\sin t\,\bar{i} - 3\cos t\,\bar{j}$

$|\bar{v}(t)| = \sqrt{9\sin^2 t + 9\cos^2 t} = \sqrt{9(1)} = 3$

$\int_C (2yx^2 - 4x)\,ds = \int_0^{\pi} [2(-3\sin t)\,9\cos^2 t - 4(3\cos t)]\,|\bar{v}(t)|\,dt$

$= \int_0^{\pi} (-54\sin t\cos^2 t - 12\cos t)\,3\,dt$

$= -6 \cdot 3\int_0^{\pi} (9\sin t\cos^2 t + 2\cos t)\,dt$

$= -18\int_0^{\pi} 9\,\underline{\sin t}\cos^2 t\,\underline{dt} - 18\int_0^{\pi} 2\cos t\,dt$

put $\cos t = m$

$-\sin t\,dt = dm \Rightarrow \sin t\,dt = -dm$

If $t = 0 \Rightarrow m = \cos 0 = 1$

$t = \pi \Rightarrow m = \cos\pi = -1$

$= -162\int_0^{-1} m^2(-dm) - 36[\sin t]_0^{\pi}$

$= 162\left[\frac{m^3}{3}\right]_0^{-1} - 36[\sin\pi - \sin 0]$

$= 162\left[-\frac{1}{3} - 0\right] - 36\left[0-0\right]$

$= -54$

Ex. Evaluate $\int_c (1+x^3)\,dx$ where c is the right half of the circle of radius 2 with counter clockwise direction followed by the line segment from (0,2) to (-3,4)

$\Rightarrow$ Here $c = c_1 \cup c_2$

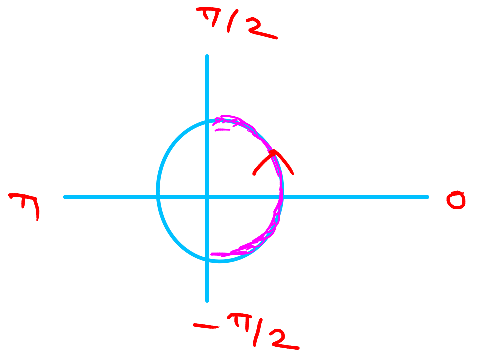
1) c_1 is right haft circle of radius 2 with counter clockwise direction

$x = a\cos t, \; y = a\sin t$

radius $a = 2$

$\therefore \; x = 2\cos t, \; y = 2\sin t$

$-\frac{\pi}{2} \le t \le \frac{\pi}{2}$



$\bar{r}(t) = x\bar{i} + y\bar{j}$

$= 2\cos t\,\bar{i} + 2\sin t\,\bar{j}$

$x = 2\cos t \Rightarrow dx = -2\sin t\,dt$

$ds \Rightarrow |\bar{r}'(t)|\,dt$

$dx = x'dt$

$dy = y'dy$

② c_2 is line from (0,2) to (-3,-4)

$\bar{r}(t) = (1-t)(0,2) + t(-3,-4)$

$= (0, 2-2t) + (-3t, -4t)$

$2-2t-4t$

$= (-3t, 2-6t)$

$\bar{r}(t) = -3t\,\bar{i} + (2-6t)\,\bar{j}$

$x = -3t, \; y = 2-6t$

$\Rightarrow dx = -3dt$

$\int_c (1+x^3)\,dx = \int_{c_1} (1+x^3)\,dx + \int_{c_2} (1+x^3)\,dx$

$= \int_{-\pi/2}^{\pi/2} [1 + 2\cos^3 t](-2\sin t)\,dt$

$$+\int_0^1 (1+(-3t)^3]\,(-3\,dt)$$

$$\int_C (1+x^3)\,dx = \int_{-\pi/2}^{\pi/2} -2\sin t\,dt - 4\int_{-\pi/2}^{\pi/2} \underbrace{\cos^3 t\cdot\sin t\,dt}_{=0}$$

$$-3\int_0^1 (1-27t^3)\,dt \quad -① $$

$$-4\int_{-\pi/2}^{\pi/2} \cos^3 t\cdot \sin t\,dt$$

put $\cos t = m$

$\Rightarrow -\sin t\,dt = dm$

If $t=-\frac{\pi}{2} \Rightarrow m = \cos(-\frac{\pi}{2}) = \cos\frac{\pi}{2} = 0$

$t = \frac{\pi}{2} \Rightarrow m = \cos(\frac{\pi}{2}) = 0$

$$-4\int_{-\pi/2}^{\pi/2} \cos^3 t\sin t\,dt = -4\int_0^0 m^3(-dm) = 0$$

$$\int_0^a f(x) = 0$$

From eqn

$$\int_C (1+x^3)\,dx = -2\left[-\cos t\right]_{-\pi/2}^{\pi/2} - 0 - 3\left[t - 27\frac{t^4}{4}\right]_0^1$$

$$= -2\left[-\cos\frac{\pi}{2} + \cos(-\frac{\pi}{2})\right] - 3\left[1 - \frac{27}{4} - 0\right]$$

$$= 0 - 3\left[\frac{-23}{4}\right]$$

$$= \frac{69}{4}$$

Ex. Find the line integral of $f(x,y) = ye^{x^2}$ along the curve $\bar{r}(t) = 4t\bar{i} - 3t\bar{j}$, $-1 \le t \le 2$

$\Rightarrow \quad \bar{v}(t) = 4\bar{i} - 3\bar{j}$

$|\bar{v}(t)| = \sqrt{16+9} = 5$

$x = 4t,\ y = -3t$ $\qquad$ $ds = |\bar{v}(t)|\,dt$

$$\int_C f(x,y)\,ds = \int_C ye^{x^2}\,ds$$

$$= \int_{-1}^{2} -3t\, e^{16t^2} |\bar{v}(t)|\, dt$$

$$= -3 \int_{-1}^{2} t\, e^{16t^2} \cdot 5\, dt$$

$$= -15 \int_{-1}^{2} e^{16t^2} \cdot t\, dt$$

put $t^2 = m$

$2t\,dt = dm \Rightarrow t\,dt = \frac{1}{2} dm$

$t = -1 \Rightarrow m = 1$ & $t = 2 \Rightarrow m = 4$

$$= -15 \int_{1}^{4} e^{16m} \frac{1}{2} dm$$

$$= \frac{-15}{2} \left[\frac{e^{16m}}{16} \right]_1^4$$

$$= \frac{-15}{32} \left[e^{64} - e^{16} \right]$$

Ex. Evaluate $\int_C 6x\, dx$ where C is the portion of $y = x^2$ from $x = -1$ to $x = 2$. The direction of C is in the direction of increasing x.

$\Rightarrow$ Take $x = t$

$\therefore y = x^2 = t^2$

$x = -1 \Rightarrow t = -1$ & $x = 2 \Rightarrow t = 2$

$x = t \Rightarrow dx = dt$

$$\int_C 6x\, dx = \int_{-1}^{2} 6(t)\, dt = 6 \left[\frac{t^2}{2} \right]_{-1}^{2} = 3[4 - 1] = 9$$

Ex. $\int_C 2y\, dx + (1 - x)\, dy$ where C is portion of $y = 1 - x^2$ from $x = -1$ to $x = 2$

$\Rightarrow$ Take $x = t$, $y = 1 - t^2$

$dx = dt$, $dy = -2t\, dt$

$x=-1 \Rightarrow t=-1$ & $x=2 \Rightarrow t=2$

$$\int_C 2y\,dx + (1-x)\,dy = \int_{-1}^{2} [2(1-t^2)\,dt + (1-t)(-2t\,dt)$$

$$= \int_{-1}^{2} [2 - \cancel{2t^2} - 2t + \cancel{2t^2}]\,dt$$

$$= \int_{-1}^{2} (2-2t)\,dt$$

$$= \left[2t - 2\frac{t^2}{2}\right]_{-1}^{2}$$

$$= (4-4) - (-2-1)$$

$$= 3$$

Ex. $\int_C \sqrt{1+y}\,dy$ where C is the portion $y=e^{2x}$ from $x=0$ to $x=2$.

$\Rightarrow$ Take $x=t$, $\Rightarrow y=e^{2t}$

$dy = 2e^{2t}\,dt$

$t=0$ to $t=2$

$$\int_C \sqrt{1+y}\,dy = \int_0^2 \sqrt{1+e^{2t}}\cdot 2e^{2t}\,dt$$

put $1+e^{2t}=m$

$\Rightarrow 2e^{2t}\,dt = dm$

$t=0 \Rightarrow m = 1+e^0 = 1+1 = 2$

$t=2 \Rightarrow m = 1+e^4$

$$= \int_2^{1+e^4} \sqrt{m}\,dm$$

$$= \int_2^{1+e^4} m^{\frac{1}{2}}\,dm = \left[\frac{m^{3/2}}{3/2}\right]_2^{1+e^4}$$

$$= \frac{2}{3}\left[(1+c^4)^{\frac{3}{2}} - 2^{3/2}\right]$$

Ex. Evaluate $\int_C \sqrt{x^2+y^2}\, ds$ along the curve

$$\bar{r}(t) = 4\cos t\, \bar{i} + 4\sin t\, \bar{j} + 3t\, \bar{k} \quad , -2\pi \le t \le 2\pi$$

$$\Rightarrow \quad \bar{v}(t) = -4\sin t\, \bar{i} + 4\cos t\, \bar{j} + 3\bar{k}$$

$$|\bar{v}(t)| = \sqrt{16+9} = \sqrt{25} = 5$$

$$x = 4\cos t, \quad y = 4\sin t, \quad z = 3t$$

$$\int_C \sqrt{x^2+y^2}\, ds = \int_{-2\pi}^{2\pi} \sqrt{16\cos^2 t + 16\sin^2 t}\; |\bar{v}(t)|\, dt$$

$$= \int_{-2\pi}^{2\pi} 4 \cdot 5\, dt$$

$$= 20\,[t]_{-2\pi}^{2\pi}$$

$$= 20\,[2\pi - (-2\pi)]$$

$$= 80\pi$$

Ex. $\int_C \frac{x^2}{y^{4/3}}\, ds$ where c is the curve $x = t^2$, $y = t^3$, $1 \le t \le 2$

$$\Rightarrow \quad \bar{r}(t) = x\bar{i} + y\bar{j} = t^2\bar{i} + t^3\bar{j}$$

$$\bar{v}(t) = 2t\,\bar{i} + 3t^2\,\bar{j}$$

$$|\bar{v}(t)| = \sqrt{4t^2 + 9t^4} = \sqrt{t^2(4+9t^2)}$$

$$= t\sqrt{4+9t^2}$$

$$\int_C \frac{x^2}{y^{4/3}}\, ds = \int_1^2 \frac{t^4}{(t^3)^{\frac{4}{3}}}\, |\bar{v}(t)|\, dt$$

$$= \int_1^2 \frac{t^4}{t^4}\, t\sqrt{4+9t^2}\, dt$$

$$= \int_1^2 \sqrt{4+9t^2} \cdot t\, dt$$

put $4+9t^2 = m$

$18t\, dt = dm$

$t\, dt = \frac{1}{18} dm$

$t=1 \Rightarrow m = 4+9 = 13$

$t=2 \Rightarrow m = 4+9(4) = 40$

$$= \int_{13}^{40} \sqrt{m} \cdot \frac{1}{18} dm$$

$$= \frac{1}{18} \left[\frac{m^{3/2}}{3/2} \right]_{13}^{40}$$

$$= \frac{1}{18} \cdot \frac{2}{3} \left[(40)^{\frac{3}{2}} - (13)^{\frac{3}{2}} \right]$$

$$= \frac{1}{27} \left[(40)^{\frac{3}{2}} - (13)^{\frac{3}{2}} \right]$$

Reference: Vector Calculus, text book for S.Y.B.Sc., by Golden series, Nirali Prakashan