K. T. S. P. Mandal's

Hutatma Rajguru Mahavidyalaya, Rajgurunagar

Tal-Khed, Dist-Pune (410 505)

Second Year Science

Semester -IV (2019 Pattern)

Subject – Vector Calculus

S. Y. B. Sc., Paper-II:MT-242(A)

Chapter 2: Integrals

Topic- Work done

Prepared by

Prof. R. M. Wayal

Department of Mathematics

Hutatma Rajguru Mahavidyalaya, Rajgurunagar

* Work done -

Let c be a smooth curve parametrized by $\bar{r}(t)$, $a \le t \le b$ and let $\bar{F}$ be a continuous force field over a region containing c. Then the work done in moving an object from the point $A = \bar{r}(a)$ to the point $B = \bar{r}(b)$ along c is

$$W = \int_c \bar{F}\cdot\bar{T}\, ds = \int_c \bar{F}(r(t))\cdot\frac{d\bar{r}}{dt}\, dt = \int_c \bar{F}\cdot d\bar{r}$$

$$\boxed{\text{Work done} = \int_c \bar{F}\cdot d\bar{r}}$$

* Flow Integrals and Circulation for velocity Fields -

If $\bar{r}(t)$ parametrizes a smooth curve c in the domain of a continuous velocity field $\bar{F}$, the field along the curve from $A = \bar{r}(a)$ to $B = \bar{r}(b)$ is

$$\boxed{\text{Flow} = \int_c \bar{F}\cdot\bar{T}\, ds = \int_c \bar{F}\cdot d\bar{r}}$$

The integral is called a flow integral. If the curve starts and ends at the same point so that $A = B$ the flow is called the circulation around the curve.

Ex. Find the work done by the force field $\bar{F} = xy\bar{i} + y\bar{j} - yz\bar{k}$ over the curve in the direction of increasing t,

$$\bar{r}(t) = t\bar{i} + t^2\bar{j} + t\bar{k}, \quad 0 \le t \le 1$$

$\Rightarrow$ $\bar{r}(t) = t\bar{i} + t^2\bar{j} + t\bar{k}$

$$\frac{d\bar{r}}{dt} = \bar{i} + 2t\bar{j} + \bar{k}$$

$$x = t,\ y = t^2,\ z = t$$

$$\bar{F}(\bar{r}(t)) = t^3\bar{i} + t^2\bar{j} - t^3\bar{k}$$

$$\bar{F}\cdot\frac{d\bar{r}}{dt} = t^3 + 2t^3 - t^3 = 2t^3$$

$$\text{Work done} = \int_c \bar{F}(r(t))\cdot\frac{d\bar{r}}{dt}\, dt = \int_c \bar{F}\cdot d\bar{r}$$

$$= \int_0^1 2t^3\, dt$$

$$= 2\left[\frac{t^4}{4}\right]_0^1$$

$$= \frac{1}{2}[1-0] = \frac{1}{2}$$

Ex. Find the work done by the force $\bar{F} = x\bar{i} + 3xy\bar{j} - (x+z)\bar{k}$ on the partical moving along the line segment that goes from $(1,4,2)$ to $(0,5,1)$

$\Rightarrow$ $\bar{r}(t) = (1-t)(1,4,2) + t(0,5,1)$, $0 \le t \le 1$

$$\bar{r}(t) = (1-t, 4-4t, 2-2t) + (0, 5t, t)$$

$$= (1-t, 4+t, 2-t)$$

$$\bar{r}(t) = (1-t)\,\bar{i} + (4+t)\,\bar{j} + (2-t)\,\bar{k}$$

$$d\bar{r} = [-1\,\bar{i} + \bar{j} - \bar{k}]\,dt$$

$$x = 1-t,\ y = 4+t,\ z = 2-t$$

$$\bar{F} = (1-t)\,\bar{i} + 3(1-t)(4+t)\,\bar{j} - [(1-t) + [2-t]]\,\bar{k}$$

$$= -\bar{i} + 3(4+t-4t-t^2)\,\bar{j} - (3-2t)\,\bar{k}$$

$$= -\bar{i} + (-3t^2 - 9t + 12)\,\bar{j} - (3-2t)\,\bar{k}$$

$$\bar{F}\cdot d\bar{r} = [1 - 3t^2 - 9t + 12 + 3 - 2t]\,dt$$

$$= (-3t^2 - 11t + 16)\,dt$$

$$\text{Work done} = \int_C \bar{F}\cdot d\bar{r}$$

$$= \int_0^1 [-3t^2 - 11t + 16]\,dt$$

$$= \left[-t^3 - 11\frac{t^2}{2} + 16t\right]_0^1$$

$$= -1 - \frac{11}{2} + 16$$

$$= 15 - \frac{11}{2} = \frac{19}{2}$$

Ex. Find the work done by the force field

$$\bar{F} = 2y\,\bar{i} + 3x\,\bar{j} + (x+y)\,\bar{k}$$ over the curve

in the direction of increasing t

$$\bar{r}(t) = \cos t\,\bar{i} + \sin t\,\bar{j} + \frac{t}{6}\,\bar{k} \quad , 0 \le t \le 2\pi$$

$\Rightarrow$ $$\bar{r}(t) = \cos t\,\bar{i} + \sin t\,\bar{j} + \frac{t}{6}\,\bar{k}$$

$$d\bar{r} = -\sin t\,dt\,\bar{i} + \cos t\,dt\,\bar{j} + \frac{1}{6}dt\,\bar{k}$$

$$x = \cos t \,,\quad y = \sin t \,,\quad z = \frac{t}{6}$$

$$\bar{F} = 2\sin t\,\bar{i} + 3\cos t\,\bar{j} + [\cos t + \sin t]\,\bar{k}$$

$$\bar{F}\cdot d\bar{r} = \left[-2\sin^2 t + 3\cos^2 t + \frac{1}{6}\cos t + \frac{1}{6}\sin t\right]dt$$

$$\text{Work done} = \int_C \bar{F}\cdot d\bar{r}$$

$$= \int_0^{2\pi} -2\sin^2 t\,dt + 3\int_0^{2\pi}\cos^2 t\,dt$$

$\sin^2 t = \frac{1-\cos 2t}{2}$

$$+ \frac{1}{6}\int_0^{2\pi}\cos t\,dt + \frac{1}{6}\int_0^{2\pi}\sin t\,dt$$

$\cos^2 t = \frac{1+\cos 2t}{2}$

$$= -2\int_0^{2\pi}\frac{1-\cos 2t}{2}\,dt + 3\int_0^{2\pi}\frac{1+\cos 2t}{2}\,dt$$

$$+ \frac{1}{6}\left[\sin t\right]_0^{2\pi} + \frac{1}{6}\left[-\cos t\right]_0^{2\pi}$$

$\cos n\pi = (-1)^n$

$\sin n\pi = 0$

$$= -\Big[t - \underbrace{\frac{\sin 2t}{2}}_{=0}\Big]_0^{2\pi} + \frac{3}{2}\Big[t + \underbrace{\frac{\sin 2t}{2}}_{=0}\Big]_0^{2\pi}$$

$$+ \frac{1}{6}\Big[\underbrace{\sin 2\pi}_{=0} - \underbrace{\sin 0}_{=0}\Big] - \frac{1}{6}\left[\cos 2\pi - \cos 0\right]$$

$$= -[2\pi] + \frac{3}{2}[2\pi] - \frac{1}{6}[1-1]$$

$$= -2\pi + 3\pi = \pi$$

Ex. Find the flow of the velocity field

$\bar{F} = y^2\,\bar{i} + 2xy\,\bar{j}$ along the following path from $(0,0)$ to $(2,4)$

ⓐ The line $y=2x$

ⓑ The parabola $y=x^2$

$\Rightarrow$ ⓐ $y=2x$

Take $x=t \Rightarrow y=2t$, $0 \le t \le 2$

$\bar{r}(t) = t\,\bar{i} + 2t\,\bar{j}$

$d\bar{r} = dt\,\bar{i} + 2dt\,\bar{j}$

$\bar{F} = 4t^2\,\bar{i} + 4t^2\,\bar{j}$

$\bar{F}.d\bar{r} = [4t^2 + 8t^2]dt = 12t^2 dt$

$$\text{Flow} = \int_C \bar{F}.d\bar{r}$$

$$= \int_0^2 12t^2 dt$$

$$= 12 \frac{t^3}{3}\Big]_0^2$$

$$= 4[8-0]$$

$$= 32$$

ⓑ $y=x^2$, $(0,0)$ to $(2,4)$

put $x=t$, $y=t^2$, $0 \le t \le 2$

$\bar{r}(t) = t\,\bar{i} + t^2\,\bar{j}$

$d\bar{r} = dt\,\bar{i} + 2t\,dt\,\bar{j}$

$\bar{F} = 4t^2\,\bar{i} + 2t^3\,\bar{j}$

$\bar{F}.d\bar{r} = [4t^2 + 4t^4]dt$

$$\text{Flow} = \int_C \bar{F}.d\bar{r}$$

$$= \int_0^2 (4t^2 + 4t^4)dt$$

$$= 4\frac{t^3}{3} + 4\frac{t^5}{5}\Big]_0^2$$

$$= \frac{4}{3}(8) + \frac{4}{5}(32)$$

$$= \frac{32}{3} + \frac{128}{5}$$

$$= \frac{160 + 384}{15} = \frac{544}{15}$$

Ex. Find the flow along the given curve in the direction of increasing t where

$\bar{F} = (x-z)\bar{i} + 0\bar{j} + x\bar{k}$ across

$\bar{r}(t) = \cos t\,\bar{i} + 0\bar{j} + \sin t\,\bar{k}$, $0 \le t \le \pi$

$\Rightarrow$ $\bar{r}(t) = \cos t\,\bar{i} + 0\bar{j} + \sin t\,\bar{k}$

$d\bar{r} = -\sin t\,dt\,\bar{i} + 0\bar{j} + \cos t\,dt\,\bar{k}$

$x = \cos t$, $y = 0$, $z = \sin t$

$\bar{F} = [\cos t - \sin t]\,\bar{i} + 0\bar{j} + \cos t\,\bar{k}$

$\bar{F}\cdot d\bar{r} = [-\sin t\cos t + \sin^2 t + \cos^2 t]\,dt$

$\sin 2\theta = 2\sin\theta\cos\theta$

$= [-\frac{1}{2}[2\sin t\cos t] + 1]\,dt$

$= (-\frac{1}{2}\sin 2t + 1)\,dt$

$$\text{Flow} = \int_C \bar{F}\cdot d\bar{r}$$

$$= \int_0^\pi (-\tfrac{1}{2}\sin 2t + 1)\,dt$$

$$= -\frac{1}{2}\left[-\frac{\cos 2t}{2}\right]_0^\pi + [t]_0^\pi$$

$$= \frac{1}{4}[\cos 2\pi - \cos 0] + [\pi - 0]$$

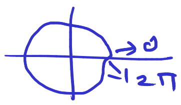
$$= \frac{1}{4}[1 - 0] + \pi$$

$$= \pi$$

Ex. Find the circulation of the field

$\bar{F} = 4\bar{i} + (x+2y)\bar{j}$ around the circle $x^2+y^2=4$.

$\Rightarrow \quad x = 2\cos t,\ y = 2\sin t \quad ,\ 0 \le t \le 2\pi$



$\bar{r}(t) = 2\cos t\,\bar{i} + 2\sin t\,\bar{j}$

$d\bar{r} = -2\sin t\,dt\,\bar{i} + 2\cos t\,dt\,\bar{j}$

$\bar{F} = 2\sin t\,\bar{i} + (2\cos t + 4\sin t)\bar{j}$

$\bar{F}\cdot d\bar{r} = [-4\sin^2 t + 4\cos^2 t + 8\sin t\cos t]\,dt$

$= 4(\cos^2 t - \sin^2 t) + 4(2\sin t\cos t)$

$= 4\cos 2t + 4\sin 2t$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

circulation $= \int_C \bar{F}\cdot d\bar{r}$

$= \int_0^{2\pi} (4\cos 2t + 4\sin 2t)\,dt$

$= 4\left[\frac{\sin 2t}{2} - \frac{\cos 2t}{2}\right]_0^{2\pi}$

$= 2\left[(\sin 4\pi - \cos 4\pi) - (\sin 0 - \cos 0)\right]$

$= 2[(0-1) - (0-1)]$

$= 2(-1+1)$

$= 0$

Ex. Find the circulation of $\bar{F} = 2x\bar{i} + 2z\bar{j} + 2y\bar{k}$ around the closed path consisting of the following curve in the direction of increasing t.

$\bar{r}(t) = \bar{j} + \frac{\pi}{2}(1-t)\bar{k} \quad ,\ 0 \le t \le 1$

$\Rightarrow \quad \bar{r}(t) = \bar{j} + \frac{\pi}{2}(1-t)\bar{k}$

$d\bar{r} = -\frac{\pi}{2}\,dt\,\bar{k}$

$x=0,\ y=1,\ k=\frac{\pi}{2}(1-t)$

$\bar{F}=0\bar{i}+\pi(1-t)\bar{j}+2\bar{k}$

$\bar{F}\cdot d\bar{r}=0+0-\pi\,dt=-\pi\,dt$

$\text{circulation}=\int_C \bar{F}\cdot d\bar{r}=\int_0^1 -\pi\,dt$

$=-\pi[t]_0^1=-\pi(1-0)=-\pi.$

* Flux across a simple closed plane curve -

If c is a smooth simple closed curve in the domain of a continuous vector field $\bar{F}=M(x,y)\bar{i}+N(x,y)\bar{j}$ in the plane and if $\bar{n}$ is the outward pointing unit normal vector on c, the flux of $\bar{F}$ across c is

$$\text{Flux of } \bar{F} \text{ across } c=\int_C \bar{F}\cdot\bar{n}\,ds$$

$$\text{Flux of } \bar{F} \text{ across } c=\int_C \bar{F}\cdot\bar{n}\,ds$$

$$=\oint M\,dy-N\,dx$$

Ex. Find the flux of the fields $\bar{F}=\overset{M}{2x}\bar{i}+(\overset{N}{x-y})\bar{j}$ accross the circle $\bar{r}(t)=a\cos t\,\bar{i}+a\sin t\,\bar{j}$ $0\le t\le 2\pi$

$\Rightarrow\quad x=a\cos t,\quad y=a\sin t$

$dx=-a\sin t\,dt,\quad dy=a\cos t\,dt$

$M\,dx-N\,dy=2x\,dx-(x-y)\,dy$

$=2a\cos t\,(-a\sin t)\,dt+[a\cos t-a\sin t]\,a\cos t\,dt$

$=\left[2a^2\sin t\cos t+a^2\cos^2 t-a^2\sin t\cos t\right]dt$

$= [-3a^2 \sin t \cos t + a^2 \cos^2 t]\, dt$

$\text{Flux} = \oint M\,dy - N\,dx$

$$= \int_0^{2\pi} [-3a^2 \sin t \cos t + a^2 \cos^2 t]\, dt$$

$$= -\frac{3}{2}a^2 \int_0^{2\pi} \sin 2t\, dt + a^2 \int_0^{2\pi} \frac{1+\cos 2t}{2}\, dt$$

$$= -\frac{3}{2}a^2 \left[-\frac{\cos 2t}{2}\right]_0^{2\pi} + \frac{a^2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= \frac{3}{4}a^2 [\cos 4t - \cos 0]$$

$$+ \frac{a^2}{2}\left[2\pi + \frac{\sin 4\pi}{2} - 0\right]$$

$$= \frac{3}{4}a^2 [1-1] + \frac{a^2}{2}(2\pi + 0)$$

$$= \pi a^2$$

Ex. Find the flux of $\overline{F} = \overset{M}{(x-y)}\,\overline{i} + \overset{N}{x}\,\overline{j}$ across the circle $x^2+y^2=1$ in the xy plane.

$\Rightarrow$ $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$

$dx = -\sin t\, dt$, $dy = \cos t\, dt$

$M = x - y$ & $N = x$

consider

$M\,dy - N\,dx = (x-y)\,dy - x\,dx$

$= [\cos t - \sin t]\cos t\, dt - \cos t(-\sin t)\, dt$

$= [\cos^2 t - \sin t \cos t + \sin t \cos t]\, dt$

$= \cos^2 t \cdot dt$

$\text{Flux} = \oint_C M\,dy - N\,dx$

$$= \int_0^{2\pi} \cos^2 t \cdot dt$$

$$= \int_0^{2\pi} \frac{1+\cos 2t}{2}\, dt$$

$$= \frac{1}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[(2\pi + 0) - (0)\right]$$

$$= \pi .$$

Reference: Vector Calculus, textbook for S.Y.B.Sc. by Golden series, Nirali Prakashan.