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Subject – Vector Calculus

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Chapter 4: Applications of Integrals

Topic- The Curl Vector Field, Stokes' Theorem, Conservative Fields and Divergence Theorem.

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Chapter-4
Applications of Integrals.

* Curl of a vector field

Let $\bar{F} = M\bar{i} + N\bar{j} + P\bar{k}$ be a vector field then curl of $\bar{F}$ is defined as

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \text{curl of } \bar{F}$$

$$\nabla \times \bar{F} = \left[\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right]\bar{i} - \left[\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right]\bar{j} + \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]\bar{k}$$

Remark - 1) $\nabla = \frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}$

2) If $f(x,y,z)$ has continuous second order partial derivatives then $\text{curl}(\nabla f) = \bar{0}$

or $\text{curl}(\text{grad } f) = \bar{0}$

3) If $\bar{F}$ is a conservative vector field then $\text{curl}(\bar{F}) = \bar{0}$.

4) If $\bar{F}$ is defined on all of $\mathbb{R}^3$ whose components have continuous first order partial derivative and $\text{curl}(\bar{F}) = \bar{0}$ then $\bar{F}$ is a conservative.

5) If $\bar{F}$ is conservative then $\oint_C \bar{F} \cdot d\bar{r} = 0$

Ex. Find the curl of vector field

$$\bar{F} = (x+y-z)\bar{i} + (2x-y+3z)\bar{j} + (3x+2y+z)\bar{k}$$

$$\Rightarrow \quad \operatorname{curl}(\bar F) = \nabla \times \bar F = \begin{vmatrix} \bar i & \bar j & \bar k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y-z & 2x-y+3z & 3x+2y+z \end{vmatrix}$$

$$= \bar i\,[2-3] - \bar j\,[3+1] + \bar k\,[2-1]$$

$$\nabla \times \bar F = -\bar i - 4\bar j + \bar k$$

2] $\bar F = x^2yz\,\bar i + xy^2z\,\bar j + xyz^2\,\bar k$

$$\Rightarrow \quad \nabla \times \bar F = \operatorname{curl}(\bar F) = \begin{vmatrix} \bar i & \bar j & \bar k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix}$$

$$= \bar i\,[xz^2 - xy^2] - \bar j\,[yz^2 - x^2y] + \bar k\,[y^2z - x^2z]$$

$$= x(z^2-y^2)\,\bar i - y(z^2-x^2)\,\bar j + z(y^2-x^2)\,\bar k$$

Ex 3] Determine whether $\bar F$ is conservative or not

$$\bar F = \left(4y^2 + \frac{3x^2y}{z^2}\right)\bar i + \left(8xy + \frac{x^3}{z^2}\right)\bar j + \left(11 - \frac{2x^3y}{z^3}\right)\bar k$$

$$\Rightarrow \quad \operatorname{curl}(\bar F) = \begin{vmatrix} \bar i & \bar j & \bar k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4y^2 + \frac{3x^2y}{z^2} & 8xy + \frac{x^3}{z^2} & 11 - \frac{2x^3y}{z^3} \end{vmatrix}$$

$$= \bar i\left[-\frac{2x^3}{z^3} + \frac{2x^3}{z^3}\right] - \bar j\left[-\frac{6x^2y}{z^3} + \frac{6x^2y}{z^3}\right]$$

$$+\bar{k}\left[8y+\frac{3x^2}{z^2}-8y-\frac{3x^2}{z^2}\right]$$

$$=\bar{0}$$

since $\text{curl}(\bar{F})=\bar{0}$

$\therefore \bar{F}$ is conservative field.

Ex. Determine whether $\bar{F}$ is conservative or not

$$\bar{F}=6x\bar{i}+(2y-y^2)\bar{j}+(6z-x^3)\bar{k}$$

$\Rightarrow$

$$\text{curl}(\bar{F})=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 6x & 2y-y^2 & 6z-x^3\end{vmatrix}$$

$$=\bar{i}[0-0]-\bar{j}[-3x^2-0]+\bar{k}[0-0]$$

$$=3x^2\bar{j}\neq 0$$

since $\text{curl}(\bar{F})\neq 0$

$\therefore \bar{F}$ is not conservative field.

Ex. P.T. $\int_C[(y+z)\bar{i}+(z+x)\bar{j}+(x+y)\bar{k}]\cdot d\bar{r}=0$

where the line integral is taken along a closed curve C.

$\Rightarrow$ Let $\bar{F}=(y+z)\bar{i}+(z+x)\bar{j}+(x+y)\bar{k}$

$$\nabla\times\bar{F}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y+z & z+x & x+y\end{vmatrix}$$

grad $=\nabla f$

$\text{div}(\bar{f})=\nabla\cdot\bar{f}$

$\text{curl}(\bar{f})=\nabla\times\bar{f}$

$$=\bar{i}[1-1]-\bar{j}[1-1]+\bar{k}[1-1]$$

$$=\bar{0}$$

Since $\text{curl}(\bar{F}) = \bar{0}$

$\therefore \bar{F}$ is conservative field

$$\therefore \oint_C \bar{F} \cdot d\bar{r} = 0$$

$$\therefore \int_C [(y+z)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}] \cdot d\bar{r} = 0.$$

Ex. Find $\text{curl}(\bar{F})$, if $\bar{F} = (x^2-y)\bar{i} + (y^2-z)\bar{j} + (z^2-x)\bar{k}$

$\Rightarrow$

$$\text{curl}(\bar{F}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y & y^2-z & z^2-x \end{vmatrix}$$

$$= \bar{i}[0+1] - \bar{j}[-1-0] + \bar{k}[0+1]$$

$$= \bar{i} + \bar{j} + \bar{k}$$

* Stoke's Thm –

Let S be a piecewise smooth oriented surface having a piecewise smooth boundary curve c. Let $\bar{F} = M\bar{i} + N\bar{j} + P\bar{k}$ be a vector field whose components have continuous 1st partial derivative on an open region containing S. Then the circulation of $\bar{F}$ around c in the direction counterclockwise with respect to the surface unit normal vector $\bar{n}$ equals the integral of the curl vector field $\nabla \times \bar{F}$ over S is

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, d\sigma$$

Ex. Use stokes thm to evaluate $\iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, d\sigma$ where $\bar{F} = z^2\bar{i} - 3xy\bar{j} + x^3y^3\bar{k}$ and S is the part of $z = 5 - x^2 - y^2$ above the plane $z = 1$. Assume that S is oriented upward.

$\Rightarrow$ surface is $z = 5 - x^2 - y^2$, $z = 1$

put $z = 1$

$$1 = 5 - x^2 - y^2$$

$$x^2 + y^2 = 4 \quad \& \quad z = 1$$

$[x^2 + y^2 = r^2$
$x = r\cos\theta,$
$y = r\sin\theta]$

Take $x = 2\cos\theta$, $y = 2\sin\theta$, $0 \le \theta \le 2\pi$

$$\therefore \bar{r} = 2\cos\theta\,\bar{i} + 2\sin\theta\,\bar{j} + \bar{k}$$

$[\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}]$

$$d\bar{r} = -2\sin\theta\, d\theta\,\bar{i} + 2\cos\theta\, d\theta\,\bar{j} + 0\bar{k}$$

$$\bar{F} = 1^2\bar{i} - 3(2\cos\theta)(2\sin\theta)\bar{j} + (2\cos\theta)^3(2\sin\theta)^3\bar{k}$$

$$\bar{F} = \bar{i} - 12\sin\theta\cos\theta\,\bar{j} + 64\cos^3\theta \cdot \sin^3\theta\,\bar{k}$$

$$\therefore \bar{F} \cdot d\bar{r} = -2\sin\theta\, d\theta - 24\sin\theta\cos^2\theta$$

$$\iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, d\sigma = \oint_C \bar{F} \cdot d\bar{r}$$

$$= \int_{\theta=0}^{2\pi} [-2\sin\theta - 24\sin\theta\cos^2\theta]\, d\theta$$

$$= 2[\cos\theta]_0^{2\pi} - 24\int_0^{2\pi} \cos^2\theta \sin\theta\, d\theta$$

put $t = \cos\theta$

$dt = -\sin\theta\, d\theta$

$\therefore \sin\theta\, d\theta = -dt$

if $\theta = 0 \Rightarrow t = 1$

if $\theta = 2\pi \Rightarrow t = 1$

$$= 2[\cos 2\pi - \cos 0] - 24\int_1^1 t^2(-dt)$$

$$= 2(1-1) - 0$$

$$= 0$$

Ex.2] Evaluate $\iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, d\bar{s}$ by stokes thm for the field $\bar{F} = y\bar{i} - x\bar{j}$ over the hemisphere $s : x^2 + y^2 + z^2 = 9$, $z \geq 0$

$\Rightarrow$ put $z = 0$ in $x^2 + y^2 + z^2 = 9$

$\therefore x^2 + y^2 = 9$

c is $x^2 + y^2 = 9$, $z = 0$

(circle with center $(0,0)$ & radius 3)

$x = 3\cos\theta$, $y = 3\sin\theta$, $z = 0$, $0 \leq \theta \leq 2\pi$

$\bar{r} = 3\cos\theta\, \bar{i} + 3\sin\theta\, \bar{j} + 0\bar{k}$

$d\bar{r} = [-3\sin\theta\, \bar{i} + 3\cos\theta\, \bar{j} + 0\bar{k}]\, d\theta$

$\bar{F} = y\bar{i} - x\bar{j} \Rightarrow \bar{F} = 3\sin\theta\, \bar{i} - 3\cos\theta\, \bar{j}$

$\bar{F} \cdot d\bar{r} = (-9\sin^2\theta - 9\cos^2\theta) = -9\, d\theta$

$$\iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, dr = \oint_C \bar{F} \cdot d\bar{r}$$

$$= \int_{\theta=0}^{2\pi} -9\, d\theta$$

$$= -9 [\theta]_0^{2\pi}$$

$$= -9(2\pi - 0)$$

$$= -18\pi.$$

Ex. Use Stokes's thm to evaluate $\iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, dr$ where $\bar{F} = y\bar{i} - x\bar{j} + yx^3\bar{k}$ and S is the portion of the sphere of radius 4 with $z \geq 0$ and the upward orientation.

$\Rightarrow$ $S: x^2 + y^2 + z^2 = 16$ [eqn of sphere with radius $r=4$ $x^2 + y^2 + z^2 = r^2$]

put $z = 0$

$\therefore x^2 + y^2 = 16$

$x = 4\cos\theta, \; y = 4\sin\theta, \; z = 0$

$\bar{r} = 4\cos\theta\,\bar{i} + 4\sin\theta\,\bar{j} + 0\,\bar{k}$

$d\bar{r} = [-4\sin\theta\,\bar{i} + 4\cos\theta\,\bar{j} + 0\,\bar{k}]\, d\theta$

$F = y\bar{i} - x\bar{j} + yx^3\bar{k}$

$\Rightarrow \bar{F} = 4\sin\theta\,\bar{i} - 4\cos\theta\,\bar{j} + 4\sin\theta(4\cos\theta)^3\bar{k}$

$\bar{F} \cdot d\bar{r} = [-16\sin^2\theta - 16\cos^2\theta + 0]\, d\theta$

$= -16\, d\theta$

$$\iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, dr = \oint_C \bar{F} \cdot d\bar{r}$$

$$= \int_0^{2\pi} -16\, d\theta = -16 [\theta]_0^{2\pi} = -32\pi.$$

* Divergence –

If $\bar{F} = M(x,y,z)\,\bar{i} + N(x,y,z)\,\bar{j} + P(x,y,z)\,\bar{k}$

then divergence of $\bar{F}$ is scalar function & is given by

$$\boxed{\nabla \cdot \bar{F} = \operatorname{div} \bar{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}}$$

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$\Rightarrow$ surface is $z = 5 - x^2 - y^2$, $z = 1$

put $z = 1$

$$1 = 5 - x^2 - y^2$$

$$x^2 + y^2 = 4 \quad \& \quad z = 1$$

$[x^2 + y^2 = r^2$, $x = r\cos\theta$, $y = r\sin\theta]$

Take $x = 2\cos\theta$, $y = 2\sin\theta$, $0 \le \theta \le 2\pi$

$$\therefore \bar{r} = 2\cos\theta\,\bar{i} + 2\sin\theta\,\bar{j} + \bar{k}$$

$[\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}]$

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$$\bar{F} = 1^2\bar{i} - 3(2\cos\theta)(2\sin\theta)\bar{j} + (2\cos\theta)^3(2\sin\theta)^3\bar{k}$$

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$$\therefore \bar{F} \cdot d\bar{r} = -2\sin\theta\, d\theta - 24\sin\theta\cos^2\theta$$

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$$= \int_{\theta=0}^{2\pi} [-2\sin\theta - 24\sin\theta\cos^2\theta]\, d\theta$$

$$= 2[\cos\theta]_0^{2\pi} - 24\int_0^{2\pi} \cos^2\theta \sin\theta\, d\theta$$

put $t = \cos\theta$

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Ex. 2] Evaluate $\iint_S (\nabla \times \bar{F}) \cdot \bar{n}\, d\bar{s}$ by stokes thm for the field $\bar{F} = y\bar{i} - x\bar{j}$ over the hemisphere $s: x^2+y^2+z^2 = 9$, $z \geq 0$

$\Rightarrow$ put $z = 0$ in $x^2+y^2+z^2 = 9$

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c is $x^2+y^2 = 9$, $z = 0$

(circle with center $(0,0)$ & radius 3)

$x = 3\cos\theta$, $y = 3\sin\theta$, $z = 0$, $0 \leq \theta \leq 2\pi$

$\bar{r} = 3\cos\theta\,\bar{i} + 3\sin\theta\,\bar{j} + 0\bar{k}$

$d\bar{r} = [-3\sin\theta\,\bar{i} + 3\cos\theta\,\bar{j} + 0\bar{k}]\, d\theta$

$\bar{F} = y\bar{i} - x\bar{j} \Rightarrow \bar{F} = 3\sin\theta\,\bar{i} - 3\cos\theta\,\bar{j}$

$\bar{F} \cdot d\bar{r} = (-9\sin^2\theta - 9\cos^2\theta) = -9\, d\theta$

$$\iint_S (\nabla\times\bar F)\cdot\bar n\,dr = \oint_C \bar F\cdot d\bar r$$
$$= \int_{\theta=0}^{2\pi} -9\,d\theta$$
$$= -9\,[\theta]_0^{2\pi}$$
$$= -9(2\pi - 0)$$
$$= -18\pi.$$

Ex. Use stoke's thm to evaluate $\iint_S (\nabla\times\bar F)\cdot\bar n\,dr$ where $\bar F = y\bar i - x\bar j + yx^3\bar k$ and S is the portion of the sphere of radius 4 with $z\ge 0$ and the upward orientation.

$\Rightarrow$ $S: x^2+y^2+z^2=16$ [eqⁿ of sphere with radius $r=4$ $x^2+y^2+z^2=r^2$]

put $z=0$

$\therefore\ x^2+y^2=16$

$x=4\cos\theta,\ y=4\sin\theta,\ z=0$

$\bar r = 4\cos\theta\,\bar i + 4\sin\theta\,\bar j + 0\,\bar k$

$d\bar r = [-4\sin\theta\,\bar i + 4\cos\theta\,\bar j + 0\,\bar k]\,d\theta$

$F = y\bar i - x\bar j + yx^3\bar k$

$\Rightarrow$ $\bar F = 4\sin\theta\,\bar i - 4\cos\theta\,\bar j + 4\sin\theta(4\cos\theta)^3\,\bar k$

$\bar F\cdot d\bar r = [-16\sin^2\theta - 16\cos^2\theta + 0]\,d\theta$

$= -16\,d\theta$

$$\iint_S (\nabla\times\bar F)\cdot\bar n\,dr = \oint_C \bar F\cdot d\bar r$$
$$= \int_0^{2\pi} -16\,d\theta = -16\,[\theta]_0^{2\pi} = -32\pi.$$

* Divergence –

If $\bar{F} = M(x,y,z)\bar{i} + N(x,y,z)\bar{j} + P(x,y,z)\bar{k}$ then divergence of $\bar{F}$ is scalar function & is given by

$$\boxed{\nabla\cdot\bar{F} = \operatorname{div}\bar{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}}$$

* Solenoidal vector field –

A vector field $\bar{F}$ is called solenoidal vector field if $\operatorname{div}\bar{F} = 0$

Ex. Find the divergence of a vector field

$\bar{F} = (x+y-z)\bar{i} + (2x-y+3z)\bar{j} + (3x+2y+z)\bar{k}$

$\Rightarrow \operatorname{div}\bar{F} = \nabla\cdot\bar{F} = \frac{\partial}{\partial x}(x+y-z) + \frac{\partial}{\partial y}(2x-y+3z) + \frac{\partial}{\partial z}(3x+2y+z)$

① curl $= \nabla\times\bar{F}$ (vector)

② div $= \nabla\cdot\bar{F}$ (scalar)

$\therefore \operatorname{div}\bar{F} = 1 + (-1) + 1 = 1$

Ex. $\bar{F} = (x^2-y)\bar{i} + (y^2-z)\bar{j} + (z^2-x)\bar{k}$

$\Rightarrow \operatorname{div}\bar{F} = \nabla\cdot\bar{F} = 2x + 2y + 2z$

Ex. $\bar{F} = \sin(xy)\bar{i} + \cos(yz)\bar{j} + \tan(xz)\bar{k}$

$\Rightarrow \operatorname{div}\bar{F} = \cos(xy)\,y - \sin(yz)\,z + \sec^2(xz)\,x$

Ex. S.T. the vector field $\bar{F} = x^2y\bar{i} + (4z^3y - xy^2)\bar{j} - z^4\bar{k}$ is a solenoidal.

$\Rightarrow \operatorname{div}\bar{F} = \cancel{2xy} + (\cancel{4z^3} - \cancel{2xy}) - \cancel{4z^3} = 0$

$\therefore \operatorname{div}\bar{F} = 0$

$\Rightarrow \bar{F}$ is solenoidal vector field.

Ex. $\bar{F} = (3x + 2z^2)\bar{i} + \frac{x^3y^2}{z}\bar{j} - (z - 7x)\bar{k}$

$\Rightarrow \text{div}\,\bar{F} = \nabla \cdot \bar{F} = 3 + \frac{2x^3y}{z} - 1 = 2 + \frac{2x^3y}{z}$

Thm - Divergence thm in three dimensions -

Let $\bar{F}$ be a vector field whose components have continuous 1st partial derivative and let S be piecewise smooth oriented closed surface. Then

$$\boxed{\iint_S \bar{F} \cdot \bar{n}\, d\sigma = \iiint_D \text{div}\,\bar{F}\, dv}$$

Ex. Find the flux of $\bar{F} = xy\bar{i} + yz\bar{j} + xz\bar{k}$ outward through the surface of the cube cut from the first octant by the planes $x=1$, $y=1$ & $z=1$

$\Rightarrow$

$$\text{Flux} = \iint_S \bar{F} \cdot \bar{n}\, d\sigma = \iiint_D \text{div}\,\bar{F}\, dv \quad \text{--- (1)}$$

(div thm)

$\bar{F} = xy\bar{i} + yz\bar{j} + xz\bar{k}$

$\text{div}\,\bar{F} = y + z + x$

$\therefore$ Flux $= \iiint_D \text{div}\,\bar{F}\, dv$

$$= \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} (x+y+z)\, dx\, dy\, dz$$

$$= \int_{z=0}^{1}\int_{y=0}^{1} \left[\frac{x^2}{2} + yx + zx\right]_{x=0}^{1} dy\, dz$$

$$= \int_{z=0}^{1}\int_{y=0}^{1} \left[\frac{1}{2} + y + z\right] dy\, dz$$

$$= \int_{z=0}^{1} \left[\frac{1}{2}y + \frac{y^2}{2} + zy \right]_{y=0}^{1} dz$$

$$= \int_{z=0}^{1} \left[\frac{1}{2} + \frac{1}{2} + z \right] dz = \int_{z=0}^{1} (1+z)\, dz$$

$$= \left[z + \frac{z^2}{2} \right]_0^1$$

$$= 1 + \frac{1}{2}$$

$$= \frac{3}{2}$$

Ex. Use divergence thm to find the outward flux of $\bar{F} = x^2\bar{i} + y^2\bar{j} + z^2\bar{k}$ across the cube cut from the first octant by the planes $x=1$, $y=1$ & $z=1$

$\Rightarrow$ $\operatorname{div} \bar{F} = 2x + 2y + 2z$

$$\text{Flux} = \iint_S \bar{F}\cdot\bar{n}\, d\sigma = \iiint_D \operatorname{div} \bar{F}\, dv$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} 2(x+y+z)\, dx\, dy\, dz$$

$$= 2 \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} (x+y+z)\, dx\, dy\, dz$$

$$= 2\left(\frac{3}{2}\right)$$

$$= 3$$

Ex. Evaluate $\iint_S \bar{F}\cdot\bar{n}\, d\sigma$, where $\bar{F} = x\bar{i} + y\bar{j} + z\bar{k}$ & S is sphere $x^2+y^2+z^2 = a^2$

$\Rightarrow$ $\bar{F} = x\bar{i} + y\bar{j} + z\vec{k}$

$\operatorname{div} \bar{F} = 1+1+1 = 3$

by divergence thm

$$\iint_S F\cdot\bar{n}\,d\sigma = \iiint_D \operatorname{div} F\,dv$$

① $\iint_R 1\,dA =$ area of R

$$= \iiint_{x^2+y^2+z^2=a^2} 3\,dv$$

② $\iiint_D 1\,dv =$ volume(D)

$$= 3\iiint_{x^2+y^2+z^2=a^2} 1\,dv$$

$$= 3\,(\text{volume of the sphere } x^2+y^2+z^2=a^2)$$

$$= 3\left(\frac{4\pi}{3}a^3\right)$$

$$= 4\pi a^3$$

Ex. S.T. $\iint_S (ax\bar{i}+by\bar{j}+cz\bar{k})\cdot\bar{n}\,d\sigma = \frac{4}{3}\pi(a+b+c)$

where S is the surface of the sphere

$x^2+y^2+z^2=1$

$\Rightarrow$ Let $\bar{F} = ax\bar{i}+by\bar{j}+cz\bar{k}$

$\operatorname{div}\bar{F} = a+b+c$

$$\iint_S F\cdot\bar{n}\,d\sigma = \iiint_D \operatorname{div}\bar{F}\,dv$$

$$= \iiint_{x^2+y^2+z^2=1} (a+b+c)\,dv$$

$$= (a+b+c)\iiint_{x^2+y^2+z^2=1} 1\,dv$$

$$= (a+b+c)\,(\text{volume of the sphere } x^2+y^2+z^2=1)$$

$= (a+b+c)\left(\frac{4}{3}\pi 1^3\right)$

$= \frac{4}{3}\pi(a+b+c)$

Ex. Find $\iint_S \bar{F}\cdot\bar{n}\,d\sigma$, where $\bar{F} = x\bar{i} - y\bar{j} + (z^2-1)\bar{k}$ and S is the cylinder formed by the surface $z=0$, $z=1$, $x^2+y^2=4$

$\Rightarrow$ $\operatorname{div}\bar{F} = 1-1+2z = 2z$

limits - $x^2+y^2=4 \Rightarrow x = r\cos\theta, y = r\sin\theta, z=z$

$dx\,dy\,dz = r\,dr\,d\theta\,dz$

(cylindrical coordinates)

$0 \le \theta \le 2\pi$, $0 \le r \le 2$, $0 \le z \le 1$

$$\iint_S \bar{F}\cdot\bar{n}\,d\sigma = \iiint_D \operatorname{div}\bar{F}\,dv$$

$$= \int_{z=0}^{1}\int_{\theta=0}^{2\pi}\int_{r=0}^{2} 2z\,r\,dr\,d\theta\,dz$$

$$= \int_{z=0}^{1}\int_{\theta=0}^{2\pi} 2z\left[\frac{r^2}{2}\right]_0^2 d\theta\,dz$$

$$= \int_{z=0}^{1}\int_{\theta=0}^{2\pi} 4z\,d\theta\,dz$$

$$= 4\int_{z=0}^{1} z\,[\theta]_0^{2\pi}\,dz$$

$$= 4(2\pi)\left[\frac{z^2}{2}\right]_0^1$$

$$= 4\pi$$

Ex. Use the divergence thm to evaluate $\iint_S \bar{F}\cdot\bar{n}\,d\sigma$ where $\bar{F} = \sin(\pi x)\bar{i} + zy^3\bar{j} + (z^2+4x)\bar{k}$ and S is the surface of the box with $-1 \le x \le 2$, $0 \le y \le 1$ & $1 \le z \le 4$. Note that all six sides of the box are included is s.

$\Rightarrow \quad \operatorname{div}\bar{F} = \pi\cos(\pi x) + 3zy^2 + 2z$

$$\iint_S \bar{F}\cdot\bar{n}\,d\sigma = \iiint_D \operatorname{div}\bar{F}\,dv$$

$$= \int_{x=-1}^{2}\int_{y=0}^{1}\int_{z=1}^{4}\left[\pi\cos(\pi x) + 3zy^2 + 2z\right]dz\,dy\,dx$$

$$= \int_{x=-1}^{2}\int_{y=0}^{1}\left[\pi\cos(\pi x)z + 3\frac{z^2}{2}y^2 + z^2\right]_{z=1}^{4} dy\,dx$$

$$= \int_{x=-1}^{2}\int_{y=0}^{1}\left[3\pi\cos(\pi x) + \frac{45}{2}y^2 + 15\right]dy\,dx$$

$$= \int_{x=-1}^{2}\left[3y\pi\cos(\pi x) + \frac{15}{2}y^3 + 15y\right]_{y=0}^{1} dx$$

$$= \int_{x=-1}^{2}\left[3\pi\cos(\pi x) + \frac{45}{2}\right]dx$$

$$= \left[3\pi\frac{\sin(\pi x)}{\pi} + \frac{45}{2}x\right]_{x=-1}^{2}$$

$$= \frac{45}{2}(2) - \frac{45}{2}(-1)$$

$$= 45 + \frac{45}{2}$$

$$\iint_S \bar{F}\cdot\bar{n}\,d\sigma = \frac{135}{2}.$$

Reference: A textbook of S.Y.B.Sc., Vector Calculus, Golden Series by Nirali Publication.