K. T. S. P. Mandal's

Hutatma Rajguru Mahavidyalaya, Rajgurunagar

Tal-Khed, Dist-Pune (410 505)

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Chapter 4: Linear Transformation

Topic- Kernel and Range of Linear Transformation

Prepared by

Prof. R. M. Wayal

Department of Mathematics

Hutatma Rajguru Mahavidyalaya, Rajgurunagar

* Kernel of a Linear transformation -

Let $T : V \to W$ be a linear transformation then the set of all vectors in V that T maps to 0 (zero) is called kernel of T (or null space). It is denoted by
$\ker(T)$

$$\ker(T) = \{ \bar{u} \in V \ / \ T(\bar{u}) = 0 \}$$

* Range of L.T. -

Let $T : V \to W$ be a linear transformation then the set of all vectors in W that are images under T of atleast one vector in V is called range of T. It is denoted by $R(T)$

$$R(T) = \{ \bar{w} \in W \ / \ T(\bar{u}) = \bar{w} \ , \text{ for some } \bar{u} \in V \}$$

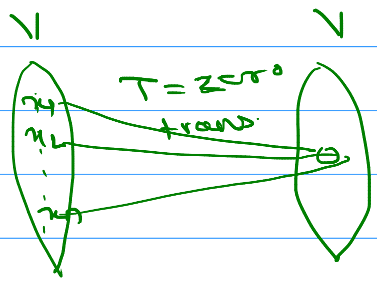
[$R(T)$ is set of all images]

Remark.

1) If $T : V \to V$ is a zero linear transformation

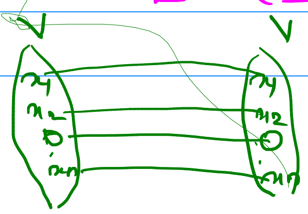
i.e. $T(\bar{u}) = \bar{0} \quad , \forall \bar{u} \in \bar{V}$

$\therefore \quad \ker(T) = V \quad \& \ R(T) = \{0\}$



2) If $I : V \to V$ is identity map i.e. $I(\bar{u}) = \bar{u}$
$\forall u \in V$.

$\ker(I) = \{\bar{0}\} \ \& \ R(I) = V$



$T : V \longrightarrow W$
ker Range

Thm Let $T : V \to W$ be linear transformation then
a) The kernel of T is a subspace of V.
b) The range of T is a subspace of W.

$\Rightarrow$ $\ker(T) = \{\bar{u} \in V \mid T(\bar{u}) = \bar{0}\}$

Let $\bar{u}, \bar{v} \in \ker(T)$

$\therefore T(\bar{u}) = \bar{0}, \quad T(\bar{v}) = \bar{0}$

① TST $\bar{u} + \bar{v} \in \ker(T)$

consider $T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v})$

$= \bar{0} + \bar{0}$

$T(\bar{u} + \bar{v}) = \bar{0}$

$\Rightarrow \bar{u} + \bar{v} \in \ker(T)$

② TST $k \cdot \bar{u} \in \ker(T)$

consider $T(k \cdot \bar{u}) = k \cdot T(\bar{u})$

$= k(\bar{0}) = \bar{0}$

$\therefore k \cdot \bar{u} \in \ker(T)$

$\therefore \ker(T)$ is subspace of V.

b) $R(T) = \{\bar{\omega} \in W \mid T(\bar{u}) = \bar{\omega}$ for some $\bar{u} \in V\}$

Let $\bar{\omega}_1, \bar{\omega}_2 \in R(T)$

$\therefore \exists\ \bar{u}_1, \bar{u}_2 \in V$ s.t.

$T(\bar{u}_1) = \bar{\omega}_1, \quad T(\bar{u}_2) = \bar{\omega}_2$

① TST $\bar{\omega}_1 + \bar{\omega}_2 \in R(T)$

since $\bar{u}_1, \bar{u}_2 \in V, \Rightarrow \bar{u}_1 + \bar{u}_2 \in V$

$\therefore T(\bar{u}_1 + \bar{u}_2) \in R(T)$

consider

$T(\bar{u}_1 + \bar{u}_2) = T(\bar{u}_1) + T(\bar{u}_2)$

$T(\bar{u}_1 + \bar{u}_2) = \bar{\omega}_1 + \bar{\omega}_2$

$\therefore \bar{\omega}_1 + \bar{\omega}_2 \in R(T)$

(2) for $\bar{u}_1 \in V$, $k\bar{u}_1 \in V \Rightarrow T(k\bar{u}_1) = R(T)$

consider $T(k\bar{u}_1) = kT(\bar{u}_1)$

$$T(k\bar{u}_1) = k\bar{\omega}_1$$

$$\therefore k\bar{\omega}_1 \in R(T)$$

* Rank –

Let $T: V \to W$ be a L.T. The dimension of range of T is called the rank of T and it is denoted by rank(T)

* Nullity –

The dimension of kernel of T is called the nullity of T and it is denoted by nullity(T)

Thm – Let the linear transformation $T: V \to W$ is injective (1-1) and $\{\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_k\}$ be a set of linearly independent vectors in V. Then $\{T(\bar{v}_1), T(\bar{v}_2), \cdots, T(\bar{v}_k)\}$ also linearly independent.

$\underbrace{T(\bar{u})}_{\in \ker T} = \bar{0} \Rightarrow \bar{u} \in \ker(T)$

Proof – consider

$$c_1 T(\bar{v}_1) + c_2 T(\bar{v}_2) + \cdots + c_k T(\bar{v}_k) = \bar{0}$$

$$T(c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_k\bar{v}_k) = \bar{0}$$

$$\Rightarrow c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_k\bar{v}_k \in \ker(T)$$

since T is injective

$$\therefore \ker(T) = \{\bar{0}\}$$

$$\therefore c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_k\bar{v}_k = \bar{0}$$

since $\{\bar{v}_1, \bar{v}_2, \cdots \bar{v}_k\}$ are linearly independent

$$\therefore c_1 = c_2 = \cdots = c_k = 0$$

$\therefore \{T(\bar{v}_1), T(\bar{v}_2), \cdots, T(\bar{v}_k)\}$ are independent.

Remark – If T is injective (one-one) then $\ker(T) = \{\bar{0}\}$

2) If $T: V \to W$ is surjective (onto) then
$Range(T) = W$

* Matrix transformation-

Let A be a fixed $m \times n$ matrix with real entries as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is any vector in $\mathbb{R}^n$. Then AX is a vector of $\mathbb{R}^m$.

Define a function $T_A : \mathbb{R}^n \to \mathbb{R}^m$ as

$$T = T_A(X) = AX = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Then T is a linear transformation, called multiplication by A. such linear transformat^n are called as matrix transformation.

Ex. Let $A = \begin{bmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix}$. Let $T: \mathbb{R}^5 \to \mathbb{R}^2$ be a L.T. such $T(x) = Ax$. Compute $T(1,0,-1,3,0)$.

$\Rightarrow$ $T(x) = Ax$

$$T(1,0,-1,3,0) = \begin{bmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1+0-1+9+0 \\ 0+0-2-3+0 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ -5 \end{bmatrix}$$

Thm- Let $T: \mathbb{R}^n \to \mathbb{R}^m$ is multiplication by an $m \times n$ matrix A. Then

a) The kernel of T is the null space of A.

b) The range of T is the column space of A.

Thm- Let $T: \mathbb{R}^n \to \mathbb{R}^m$ is multiplication by an $m \times n$ matrix A, then

a) $\text{nullity}(T) = \text{nullity}(A)$

b) $\text{rank}(T) = \text{rank}(A)$.

Ex. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator given by the formula

$$T(x,y) = (2x-y, -8x+4y)$$

i) Which of the following vectors are in $\ker(T)$?

ⓐ $(5,10)$, ⓑ $(3,2)$ ⓒ $(1,1)$

ii) Which of the following vectors are in $R(T)$?

ⓐ $(1,-4)$ ⓑ $(5,0)$ ⓒ $(-3,12)$

$\Rightarrow$ i) $\ker(T)$, $T(x,y) = (2x-y, -8x+4y)$

ⓐ $T(5,10) = (10-10, -40+40) = (0,0)$

$\therefore (5,10) \in \ker(T)$

ⓑ $T(3,2) = (6-2, -24+8) = (4,-16) \neq (0,0)$

$\therefore (3,2) \notin \ker(T)$

ⓒ $T(1,1) = (2-1, -8+4) = (1,-4) \neq (0,0)$

$\therefore (1,1) \notin \ker(T)$

ii) If $(a,b) \in R(T)$ then $\exists (x,y) \in \mathbb{R}^2$ s.t.

$$T(x,y) = (a,b)$$

$$(2x-y, -8x+4y) = (a,b)$$

$$2x - y = a$$
$$-8x + 4y = b$$

[If it has sol^n then $(a,b) \in R(T)$]

$$\left[\begin{array}{cc:c} 2 & -1 & a \\ -8 & 4 & b \end{array}\right]$$

$\frac{1}{2}R_1$

$$\sim \left[\begin{array}{cc:c} 1 & -\frac{1}{2} & a/2 \\ -8 & 4 & b \end{array}\right]$$

$R_2 + 8R_1$

$$\sim \left[\begin{array}{cc:c} 1 & -\frac{1}{2} & \frac{a}{2} \\ 0 & 0 & b+4a \end{array}\right]$$

It has soln if $b + 4a = 0$ i.e. $4a + b = 0$

a) $(1,-4) = (a,b)$

$4a + b = 4(1) - 4 = 0$

$\therefore (1,-4) \in R(T)$

b) $(a,b) = (5,0)$

$4a + b = 20 + 0 = 20 \neq 0$

$\therefore (5,0) \notin R(T)$

c) $(-3,12) = (a,b)$

$4a + b = 4(-3) + 12 = 0$

$\therefore (-3,12) \in R(T)$.

Ex. Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the L.T. given by the formula

$$T(x,y,z) = (x + 6y - 2z,\ 2x - 4y + z)$$

i) Which of the following vectors are in ker(T)

a) $(1,1,-\frac{1}{2})$ b) $(\frac{3}{2}, \frac{15}{4}, 12)$ c) $(6,15,48)$

ii) Which of the following vectors are in R(T)?

a) $(5,1)$ b) $(6,-1)$ c) $(1,1)$

ker T $\Rightarrow$ $T(x,y,z) = (0,0)$

R(T) $\Rightarrow$ $T(x,y,z) = (a,b)$

$\Rightarrow$ i) $\ker(T)$, $T(x,y,z) = (x+6y-2z,\ 2x-4y+z)$

a) $T(1,1,-\frac{1}{2}) = (1+6+1,\ 2-4-\frac{1}{2}) \neq (0,0)$

$\therefore (1,1,-\frac{1}{2}) \notin \ker(T)$

b) $T(\frac{3}{2}, \frac{15}{4}, 12) = \left(\frac{3}{2} + \frac{45}{2} - 24,\ 3-15+12\right)$

$= (24-24, 0) = (0,0)$

$\therefore (\frac{3}{2}, \frac{15}{4}, 12) \notin \ker T$

c) $T(6,15,48) = (6+90-96,\ 12-60+48)$

$= (0,0)$

$\therefore (6,15,48) \in \ker(T)$

ii) If $(a,b) \in R(T)$ then

$T(x,y,z) = (a,b)$

$(x+6y-2z,\ 2x-4y+z) = (a,b)$

$x+6y-2z = a$

$2x-4y+z = b$

$$\left[\begin{array}{ccc|c} 1 & 6 & -2 & a \\ 2 & -4 & 1 & b \end{array}\right]$$

$R_2 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 6 & -2 & a \\ 0 & -16 & 5 & b-2a \end{array}\right]$$

$-\frac{1}{16} R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 6 & -2 & a \\ 0 & 1 & -\frac{5}{16} & \frac{b-2a}{16} \end{array}\right]$$

$\text{rank}(A) = 2$ & $\text{rank}(A|B) = 2$

$\therefore$ This system alway has sol^n

for $(a,b) \in \mathbb{R}^2$

$\therefore (a,b) \in R(T)$ for every a & b

$\therefore (5,1), (6,-1), (1,1) \in R(T)$

Ex. $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the L.T. given by the formula

$$T(x,y,z) = (x+y-z, x-2y+z, -2x-2y+2z)$$

i) Which of the following vectors are in $\ker(T)$?

a) $(1,2,3)$, b) $(1,2,1)$ c) $(-1,1,2)$.

ii) Which of the following vectors are in $R(T)$?

a) $(1,2,-2)$ b) $(3,5,2)$ c) $(-2,3,4)$

$\Rightarrow$ a) $T(1,2,3) = (0,0,0)$

$\therefore (1,2,3) \in \ker(T)$

b) $T(1,2,1) = (1+2-1, 1-4+1, -2-4+2) = (2,-2,-4)$

$\neq (0,0,0)$

$\therefore (1,2,1) \notin \ker(T)$

c) $T(-1,1,2) = (-1+1-2, -1-2+2, 2-2+4)$

$\neq (0,0,0)$

$\therefore (-1,1,2) \notin \ker(T)$

ii) $(a,b,c) \in R(T)$ if

$$T(x,y,z) = (a,b,c)$$

$$(x+y-z, x-2y+z, -2x-2y+2z) = (a,b,c)$$

$$x+y-z = a$$

$$x-2y+z = b$$

$$-2x-2y+2z = c$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & a \\ 1 & -2 & 1 & b \\ -2 & -2 & 2 & c \end{array}\right]$$

$$R_2-R_1,\ R_3+2R_1 \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & a \\ 0 & -3 & 2 & b-a \\ 0 & 0 & 0 & c+2a \end{array}\right]$$

$-\frac{1}{3}R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & a \\ 0 & 1 & -\frac{2}{3} & \frac{-b+a}{3} \\ 0 & 0 & 0 & c+2a \end{array}\right]$$

This system has soln if $c+2a=0$ i.e. any vector $(a,b,c) \in R(T)$ if it satisfy

$c+2a=0$

(a) $(1,2,-2)$

$c+2a = -2+2(1) = 0$

$\therefore (1,2,-2) \in R(T)$

(b) $(3,5,2)$

$c+2a = 2+6 = 8 \neq 0$

$\therefore (3,5,2) \notin R(T)$

(c) $(-2,3,4)$

$c+2a = 4+2(-2) = 0$

$\therefore (-2,3,4) \in R(T)$

Ex. Find the basis for kernel & and range of linear transformation.

a) $T: \mathbb{R}^3 \to \mathbb{R}^3$ given by the formula

$T(x,y,z) = (x+y+2z,\ x+z,\ 2x+y+3z)$

$\Rightarrow$ $\ker(T)$ –

consider $T(x,y,z) = 0$

$(x+y+2z,\ x+z,\ 2x+y+3z) = (0,0,0)$

$x+y+2z = 0$

$x+z = 0$

$2x+y+3z = 0$

$$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 \end{array}\right]$$

$R_2 - R_1,\ R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$-R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$R_3 + R_2$

$$\begin{array}{cccc} x & y & z & = \end{array}$$
$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = 2 = \rho(A|B) <$ no. of variables

$\therefore$ It has infinitely many soln

Take z as a free variable

$z = t, \quad t \in \mathbb{R}$

$x + y + 2z = 0 \quad -①$

$y + z = 0 \quad \Rightarrow y = -t$

from ①

$x - t + 2t = 0$

$x = -t$

$\ker(T) = \{(-t, -t, t) \mid t \in \mathbb{R}\}$

$= \{t(-1, -1, 1) \mid t \in \mathbb{R}\}$

Basis for $\ker(T) = \{(-1, -1, 1)\}$

$\therefore$ dim of $\ker(T) = 1$

Nullity $(T) = 1$

Basis for Range(T) –

Basis for range of T is column space of A.

$$Tx = Ax$$

where

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

Basis of column space of A = Basis for row space A^T

consider

$$A^T = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$ [Interchange row & column]

reduce this A^T to row echelon form

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ (by using above calculation)

Basis for row space of $A^T = \{(1,1,2),(0,1,1)\}$

$\therefore$ Basis for column space of $A = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

Basis for range of $T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

$$\dim R(T) = 2$$
$$\text{rank}(T) = 2.$$

OR

[To find basis for col^n space reduced the matrix A to reduced row echelon form & column corresponding to leading 1's form a basis for column space of A]

Ex. $T : \mathbb{R}^3 \to \mathbb{R}^4$

$$T(x,y,z) = (x+2y+3z, -2x-y, x-2y-5z, 4y+8z)$$

$\Rightarrow$ ker(T)

consider $T(x,y,z) = \bar{0}$

$$(x+2y+3z, -2x-y, x-2y-5z, 4y+8z) = (0,0,0,0)$$

$$x+2y+3z = 0$$
$$-2x-y = 0$$
$$x-2y-5z = 0$$
$$4y+8z = 0$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 0 \\ -2 & -1 & 0 & 0 \\ 1 & -2 & -5 & 0 \\ 0 & 4 & 8 & 0 \end{array}\right]$$

$R_2 + 2R_1$, $R_3 - R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 3 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & -4 & -8 & 0 \\ 0 & 4 & 8 & 0 \end{array}\right]$$

$\frac{1}{3} R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -4 & -8 & 0 \\ 0 & 4 & 8 & 0 \end{array}\right]$$

$R_3 + 4R_2$, $R_3 - 4R_2$

$$\begin{array}{c} \begin{array}{cccc} x & y & z & = \end{array} \\ \sim \left[\begin{array}{ccc:c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Take $z = t, \quad t \in \mathbb{R}$

$x + 2y + 3z = 0 \Rightarrow x - 4t + 3t = 0 \Rightarrow x = t$

$y + 2z = 0 \Rightarrow y = -2t$

$\ker(T) = \{ (t, -2t, t) \mid t \in \mathbb{R} \}$

$= \{ t(1, -2, 1) \mid t \in \mathbb{R} \}$

$\therefore$ Basis for $\ker(T) = \{ (1, -2, 1) \}$

$\dim \ker(T) = 1$

$\therefore$ nullity$(T) = 1$.

* Basis for Range(T) = Basis for column space of A

where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -1 & 0 \\ 1 & -2 & -5 \\ 0 & 4 & 8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Leading ones are corresponding to 1^{st} & 2^{nd} column. Therefore 1^{st} & 2^{nd} form a basis for column space of A.

Basis for col^n space $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

or you can write 1^{st} & 2^{nd} column of A

i.e. $= \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -2 \\ 4 \end{bmatrix} \right\}$

$\therefore$ Basis for $R(T) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\dim R(T) = 2$

rank $(T) = 2$

Ex. In the following case. Let be multiplication by matrix A.

a) Find basis for range of T.

b) —"— kernel of T

c) rank & nullity of T.

i) $A = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 2 & 6 & -8 & -6 \end{bmatrix}$

$\Rightarrow$ Basis for Range(T) = Basis for col^n space of A.

$$A = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 2 & 6 & -8 & -6 \end{bmatrix}$$

$$\xrightarrow{R_2 - 2R_1} \sim \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 0 & -16 & -16 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{16}R_2} \sim \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_1 - 4R_2$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Leading 1's are corresponding to 1^{st} & 3^{rd} column. Therefore 1^{st} & 3^{rd} column form a basis for column space of A.

$\therefore$ Basis for col^n sp. of $A = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\therefore$ Basis for Range$(T) = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$\therefore$ rank$(T) = 2$

* Basis for ker(T)

consider, $TX = 0$, since $TX = AX$

$\therefore AX = 0$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$
$$\begin{bmatrix} 1 & 3 & 4 & 5 \\ 2 & 6 & -8 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc:c} 1 & 3 & 4 & 5 & 0 \\ 2 & 6 & -8 & -6 & 0 \end{array}\right]$$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & = \end{array}$$
$$\sim \left[\begin{array}{cccc:c} 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Take x_2 & x_4 as a free variables

$x_2 = s$, $x_4 = t$, $s, t \in \mathbb{R}$

$x_1 + 3x_2 + x_4 = 0$ —①

$x_3 + x_4 = 0$

$x_3 = -t$

$$x_4 + 3s + t = 0$$
$$x_4 = -3s - t$$

$\therefore \ker(T) = \{(-3s-t, s, -t, t) \mid s, t \in \mathbb{R}\}$

$= \{(-3s, s, 0, 0) + (-t, 0, -t, t) \mid s, t \in \mathbb{R}\}$

$= \{s(-3, 1, 0, 0) + t(-1, 0, -1, 1) \mid s, t \in \mathbb{R}\}$

$\therefore$ Basis for $\ker(T) = \{(-3, 1, 0, 0), (-1, 0, -1, 1)\}$

$\dim \ker(T) = 2$

$\therefore$ nullity$(T) = 2$

ii) $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 1 \\ 5 & -1 & 5 \end{bmatrix}$

$\Rightarrow$ Basis for Range(T) = Basis for column space of A

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 1 \\ 5 & -1 & 5 \end{bmatrix}$$

$R_2 - 3R_1,\ R_3 - 5R_1$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -5 \\ 0 & 4 & -5 \end{bmatrix}$$

$\frac{1}{4}R_2$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -5/4 \\ 0 & 4 & -5 \end{bmatrix}$$

$R_1 + R_2,\ R_3 - 4R_2$

$$\sim \begin{bmatrix} 1 & 0 & \frac{3}{4} \\ 0 & 1 & -5/4 \\ 0 & 0 & 0 \end{bmatrix}$$

leading 1s are corresponding to 1^{st} & 2^{nd} col^n
$\therefore$ 1^{st} & 2^{nd} col^n form a basis for col^n space of A.

Basis for col^n sp. $A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\therefore$ Basis for range of $T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$$\text{rank}(T) = 2$$

* ker(T)

consider, $TX = 0$ i.e. $AX = 0$

$$\left[\begin{array}{ccc:c} 1 & -1 & 2 & 0 \\ 3 & 1 & 1 & 0 \\ 5 & -1 & 5 & 0 \end{array}\right]$$

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & = \end{array} \\ \sim \left[\begin{array}{ccc:c} 1 & 0 & 3/4 & 0 \\ 0 & 1 & -5/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Take x_3 as a free variable

$$x_3 = t, \quad t \in \mathbb{R}$$

$$x_1 + \frac{3}{4} x_3 = 0 \Rightarrow x_1 = -\frac{3}{4} t$$

$$x_2 - \frac{5}{4} x_3 = 0 \Rightarrow x_2 = \frac{5}{4} t$$

$\therefore$ ker(T) $= \left\{ \left(-\frac{3}{4}t, \frac{5}{4}t, t\right) \mid t \in \mathbb{R} \right\}$
(or null space)

$$= \left\{ t\left(-\frac{3}{4}, \frac{5}{4}, 1\right) \mid t \in \mathbb{R} \right\}$$

$\therefore$ Basis for ker(T) $= \left\{ \left(-\frac{3}{4}, \frac{5}{4}, 1\right) \right\}$

$\therefore$ nullity $= 1$

* Dimension thm for L.T.

If $T : V \to W$ be a L.T. then
$$\text{rank}(T) + \text{nullity}(T) = \dim(V)$$

Ex. In each part find nullity of T

a) $T : \mathbb{R}^5 \to \mathbb{R}^7$ has rank 3.

$\Rightarrow$ $\text{rank}(T) + \text{nullity}(T) = \dim(\mathbb{R}^5)$

$3 + \text{nullity}(T) = 5$

$\Rightarrow \text{nullity}(T) = 2$

b) $T : P_4 \to P_3$, rank $= 1$

$\text{rank}(T) + \text{nullity}(T) = \dim(P_4)$

$1 + \text{nullity}(T) = 5$

$\text{nullity}(T) = 4$.

c) The range of $T : \mathbb{R}^6 \to \mathbb{R}^3$ is $\mathbb{R}^3$ then find nullity of T.

$\Rightarrow$ range of $T = \mathbb{R}^3$

$\Rightarrow \text{rank}(T) = 3$

$\text{rank}(T) + \text{nullity}(T) = \dim(\mathbb{R}^6)$

$3 + \text{nullity}(T) = 6$

$\Rightarrow \text{nullity}(T) = 3$

d) $T : M_{22} \to M_{22}$ has rank 3.

$\text{rank}(T) + \text{nullity}(T) = \dim(M_{22})$

$3 + \text{nullity}(T) = 4$

$\text{nullity}(T) = 1$

Thm - Let $T : V \to W$ is linear transformation from n-dimensional vector space V to vector space W, then the range of T is finite-dimensional and

$$rank(T)+nullity(T)=n$$

or

$$\dim R(T)+\dim \ker(T)=\dim V.$$

Proof - Let $\dim \ker(T)=r$

where $\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_r\}$ is basis of $\ker(T)$.

since $\ker(T)$ is subspace of V

$\therefore$ $\dim \ker(T) \le \dim V$ or $nullity(T) \le \dim V$

i.e. $r \le n$

since $\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_r\}$ is basis for $\ker(T)$

$\therefore$ It is linearly independent in V.

$\therefore$ It can be extended to basis for V.

say

$$\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_r, \bar{v}_{r+1}, \bar{v}_{r+2}, \dots, \bar{v}_{r+k}\}$$

since $\dim V=n$

$$\therefore r+k=n \quad — ①$$

claim - TST $S=\{T(\bar{v}_{r+1}), T(\bar{v}_{r+2}), \dots, T(\bar{v}_{r+k})\}$ is basis for range of T.

① We show that S is L.I.

consider

$$c_1T(\bar{v}_{r+1})+c_2T(\bar{v}_{r+2})+\dots+c_kT(\bar{v}_{r+k})=\bar{0}$$

$$T(c_1\bar{v}_{r+1}+c_2\bar{v}_{r+2}+\dots+c_k\bar{v}_{r+k})=\bar{0}$$

$$\Rightarrow c_1\bar{v}_{r+1}+c_2\bar{v}_{r+2}+\dots+c_k\bar{v}_{r+k} \in \ker(T)$$

since $\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_r\}$ is basis for $\ker(T)$

$$\therefore c_1\bar{v}_{r+1}+\dots+c_k\bar{v}_{r+k}=\alpha_1\bar{v}_1+\alpha_2\bar{v}_2+\dots+\alpha_r\bar{v}_r$$

$$\therefore \alpha_1\bar{v}_1+\dots+\alpha_r\bar{v}_r+(-c_1)\bar{v}_{r+1}+\dots+(-c_k)\bar{v}_{r+k}=\bar{0}$$

since $\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_{r+k}\}$ is basis for V

$\therefore$ It is linearly independent.

$\therefore \alpha_1 = \cdots = \alpha_r = 0$ & $c_1 = c_2 = \cdots = c_k = 0$

i.e. $c_1 = c_2 = \cdots = c_k = 0$

$\Rightarrow$ S is linearly independent.

② TST S span R(T)

Let $\bar{w} \in R(T)$ then $\exists\, \bar{u} \in V$ s.t.

$$T(\bar{u}) = \bar{w}$$

since $\bar{u} \in V$ & $\{\bar{v}_1, \dots \bar{v}_r, \bar{v}_{r+1}, \dots, \bar{v}_{r+k}\}$ is basis for V

$\therefore \bar{u} = \alpha_1 \bar{v}_1 + \cdots + \alpha_r \bar{v}_r + \alpha_{r+1} \bar{v}_{r+1} + \cdots + \alpha_{r+k} \bar{v}_{r+k}$

$\therefore T(\bar{u}) = \alpha_1 T(\bar{v}_1) + \cdots + \alpha_r T(\bar{v}_r) + \alpha_{r+1} T(\bar{v}_{r+1}) + \cdots + \alpha_{r+k} T(\bar{v}_{r+k})$

$\bar{w} = 0 + \cdots + 0 + \alpha_{r+1} T(\bar{v}_{r+1}) + \cdots + \alpha_{r+k} T(\bar{v}_{r+k})$

since $\bar{v}_1, \dots, \bar{v}_r \in \ker(T)$

$\therefore T(\bar{v}_1) = \cdots = T(\bar{v}_r) = 0$

$\therefore \bar{w} = \alpha_{r+1} T(\bar{v}_{r+1}) + \cdots + \alpha_{r+k} T(\bar{v}_{r+k})$

$\therefore$ S span R(T)

$\therefore$ S is basis for R(T)

& $\dim R(T) = k$

From ①

$$r + k = n$$

$$\text{nullity}(T) + \text{rank}(T) = n$$

i.e. $\text{rank}(T) + \text{nullity}(T) = n$

or $\dim R(T) + \dim \ker(T) = \dim V.$

Ex. $T : \mathbb{R}^4 \to \mathbb{R}^3$ given by the formula

$T(x_1, x_2, x_3, x_4) = (x_1 - x_2 + 2x_3 + x_4,\ x_1 - x_2 + 3x_3 + 2x_4,$
$2x_1 - 2x_2 + 4x_3 + 2x_4)$

then find bases for range and kernel of T. Also determine rank and nullity of T & verify dimension thm.

$\Rightarrow$ ① R(T)

$TX = AX$

where

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 1 & -1 & 3 & 2 \\ 2 & -2 & 4 & 2 \end{bmatrix}$$

$R_2 - R_1,\ R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 - 2R_2$

$$\sim \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading ones corresponding to 1st & 3rd column. Therefore 1st & 3rd columns form basis for R(T)

$\therefore$ Basis for $R(T) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\therefore$ rank$(T) = 2$

② Nullity -

consider $AX = 0$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 1 & -1 & 3 & 2 & 0 \\ 2 & -2 & 4 & 2 & 0 \end{array}\right]$$

$$\sim \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & = \\ \end{array}$$
$$\left[\begin{array}{cccc:c} 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Take $x_2 = s$, $x_4 = t$, $s, t \in \mathbb{R}$

$$x_1 - x_2 - x_4 = 0$$
$$x_1 = s - t$$

$$x_3 + x_4 = 0$$
$$x_3 = -t$$

$\therefore \ker(T) = \{(s-t, s, -t, t) / s, t \in \mathbb{R}\}$
$= \{s(1,1,0,0) + t(-1,0,-1,1) / s, t \in \mathbb{R}\}$

$\therefore$ Basis for $\ker(T) = \{(1,1,0,0), (-1,0,-1,1)\}$

$\therefore$ nullity $(T) = 2$

$\therefore$ rank(T) + nullity$(T) = 2+2 = 4 = \dim(\mathbb{R}^4)$

$\therefore$ dimension thm verified.

Reference: A textbook of S.Y.B.Sc., Linear Algebra by Golden Series, Nirali Publication.