K. T. S. P. Mandal's

Hutatma Rajguru Mahavidyalaya, Rajgurunagar

Tal-Khed, Dist-Pune (410 505)

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Chapter 4: Linear Transformation

Topic- Composite Function

Prepared by

Prof. R. M. Wayal

Department of Mathematics

Hutatma Rajguru Mahavidyalaya, Rajgurunagar

* Composite Transformation –

* Domain, codomain and range of linear transformation

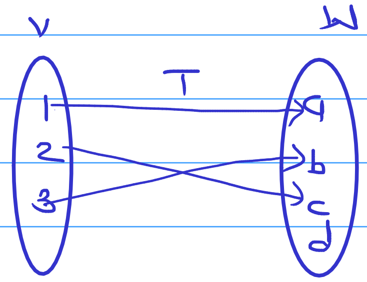
If $T : V \to W$ is linear transformation then V is called domain of T and W is called codomain of T, while $T(V)$ is range of linear transformation T.

* One-to-one Linear Transformation (Injective)

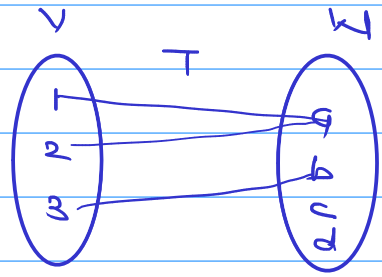
A transformation $T : V \to W$ is one-to-one (injective) if distinct elements of V have distinct images.

i.e. if $\bar{u} \neq \bar{v}$

$T(\bar{u}) \neq T(\bar{v})$



1-1

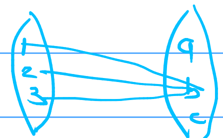


not one-one.

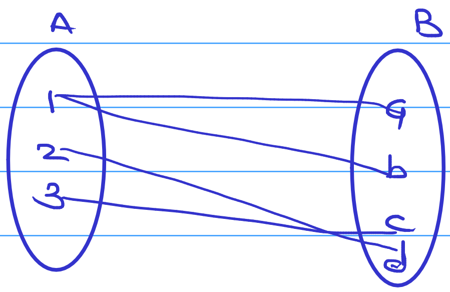
[TST T is one-one

consider $T(\bar{u}) = T(\bar{v})$

& s.t. $\bar{u} = \bar{v}$]



This is f^n



not f^n

[∵ 2 has no image]

$f : A \to B$, not function [since 1 has 2 images]

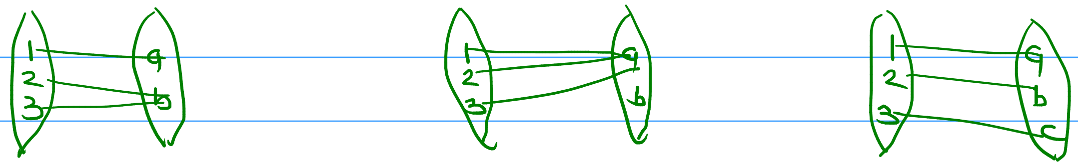
f is function if every element in A has unique image in B

* Onto Linear Transformation –

A transformation $T : V \to W$ is said to be onto (surjective) if for every $\bar{w} \in W$ $\exists$ $\bar{v} \in V$ s.t.

$$T(\bar{v}) = \bar{w}$$

i.e. every vector in W has preimage in V.
In this case $R(T) = W$



onto not onto 1-1 & onto

* Bijective –

A L.T. which is both one-one and onto is called bijective.

* Isomorphism –

A bijective linear transformation is called isomorphism.

Thm – A linear transformation $T : V \to W$ is injective (1-1) if & only if $\ker(T) = \{\bar{0}\}$ i.e. if & only if $\text{null}(T) = \{\bar{0}\}$.

Proof – 1^{st} consider $T : V \to W$ is injective

let $\bar{u} \in \ker(T)$

$\therefore T(\bar{u}) = \bar{0}$

we know that $T(\bar{0}) = \bar{0}$ (always)

$\therefore T(\bar{u}) = T(\bar{0})$

since T is 1-1

$\therefore \bar{u} = \bar{0}$

$\therefore \ker(T) = \{\bar{0}\}$

Conversely, consider $\ker(T) = \{\bar{0}\}$

TST T is one-one

consider $T(\bar{u}) = T(\bar{v})$

$$T(\bar{u}) - T(\bar{v}) = \bar{0}$$

$$T(\bar{u} - \bar{v}) = \bar{0}$$

$$\Rightarrow \bar{u} - \bar{v} \in \ker(T) = \{\bar{0}\}$$

$$\Rightarrow \bar{u} - \bar{v} = \bar{0}$$

$$\Rightarrow \bar{u} = \bar{v}$$

$\therefore T$ is one-one (injective)

* Definition – Let $T_1 : U \to V$ and $T_2 : V \to W$ be two linear transformations then the composite transformation of two transformations T_1 & T_2 is denoted by $T_2 \circ T_1 : U \to W$ and is defined as follows

$$(T_2 \circ T_1)(\bar{u}) = T_2[T_1(\bar{u})] \quad , \forall \bar{u} \in V.$$

Thm – Let $T_1 : U \to V$ & $T_2 : V \to W$ be two linear transformations then the composite transformation $T_2 \circ T_1 : U \to W$ is also a linear transformation.

Proof – Let $\bar{u}, \bar{v} \in U$ & k is any scalar

① consider $(T_2 \circ T_1)(\bar{u} + \bar{v}) = T_2[T_1(\bar{u} + \bar{v})]$

$$= T_2[T_1(\bar{u}) + T_1(\bar{v})]$$

$$= T_2[T_1(\bar{u})] + T_2[T_1(\bar{v})]$$

$$= (T_2 \circ T_1)(\bar{u}) + (T_2 \circ T_1)(\bar{v})$$

② consider

$$(T_2 \circ T_1)(k\bar{u}) = T_2[T_1(k\bar{u})]$$

$$= T_2[k T_1(\bar{u})]$$

$$= k T_2[T_1(\bar{u})]$$

$$= k (T_2 \circ T_1)(\bar{u})$$

$\therefore T_2 \circ T_1$ is linear transformation.

Ex. Find domain and codomain of $T_2 \circ T_1$ further find $(T_2 \circ T_1)(x,y)$ if

a) $T_1(x,y) = (2x, 3y)$, $T_2(x,y) = (x-y, x+y)$

$\Rightarrow$ $T_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $T_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\underset{\substack{2^{nd} \\ C}}{T_2} \circ \underset{\substack{1^{st} \\ B}}{T_1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$F : \underset{\text{domain}}{A} \longrightarrow \underset{\text{codomain}}{B}$

$\therefore$ Domain of $T_2 \circ T_1$ is $\mathbb{R}^2$

codomain of $T_2 \circ T_1$ is $\mathbb{R}^2$

$T_2(x,y) = (x-y, x+y)$

$$(T_2 \circ T_1)(x,y) = T_2[T_1(x,y)]$$
$$= T_2(2x, 3y)$$
$$= (2x-3y, 2x+3y)$$

b) $T_1(x,y) = (x-3y, 0)$, $T_2(x,y) = (4x-5y, 3x-6y)$

$\Rightarrow$

$T_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $T_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\underset{2}{T_2} \circ \underset{1}{T_1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

domain of $T_2 \circ T_1 = \mathbb{R}^2$

codomain of $T_2 \circ T_1 = \mathbb{R}^2$

$$(T_2 \circ T_1)(x,y) = T_2(T_1(x,y))$$
$$= T_2(x-3y, 0)$$
$$= (4(x-3y) - 5(0), 3(x-3y) - 6(0))$$
$$= (4x-12y, 3x-9y)$$

$$(T_1 \circ T_2)(x,y) = T_1(T_2(x,y))$$
$$= T_1(4x-5y, 3x-6y)$$
$$= (4x-5y-3(3x-6y), 0).$$

$= (-5x+13y, 0)$

c) $T_1(x,y,z) = (x-y, y+z, x-z)$

$T_2(x,y,z) = (0, x+y+z)$. Find $(T_2 \circ T_1)(x,y,z)$

$\Rightarrow$

$T_1 : \mathbb{R}^3 \to \mathbb{R}^3$, $T_2 : \mathbb{R}^3 \to \mathbb{R}^2$

$T_2 \circ T_1 : \underset{D}{\mathbb{R}^3} \to \underset{C}{\mathbb{R}^2}$

2 1

domain of $T_2 \circ T_1 = \mathbb{R}^3$

codomain of $T_2 \circ T_1 = \mathbb{R}^2$

$$(T_2 \circ T_1)(x,y,z) = T_2(T_1(x,y,z))$$
$$= T_2(x-y, y+z, x-z)$$
$$= (0, x-\cancel{y}+\cancel{y}+\cancel{z}+x-\cancel{z})$$
$$= (0, 2x)$$

Ex. Find domain & codomain of $T_3 \circ T_2 \circ T_1$ and find $(T_3 \circ T_2 \circ T_1)(x,y)$ if

$T_1(x,y) = (x+y, y-x)$, $T_2(x,y) = (0, x+y, 3y)$

$T_3(x,y,z) = (3x+2y, 4z-x-3y)$.

$\Rightarrow$

$T_1 : \mathbb{R}^2 \to \mathbb{R}^2$, $T_2 : \mathbb{R}^2 \to \mathbb{R}^3$, $T_3 : \mathbb{R}^3 \to \mathbb{R}^2$

$T_3 \circ T_2 \circ T_1 : \mathbb{R}^2 \to \mathbb{R}^2$

2 1

domain of $T_3 \circ T_2 \circ T_1 = \mathbb{R}^2$

codomain of $T_3 \circ T_2 \circ T_1 = \mathbb{R}^2$

$$(T_3 \circ T_2 \circ T_1)(x,y) = (T_3 \circ T_2)(T_1(x,y))$$

$$= (T_3 \circ T_2)(x+y,\ y-x)$$
$$= T_3(T_2(x+y,\ y-x))$$
$$= T_3(0,\ x+y+y-x,\ 3(y-x))$$
$$= T_3(0,\ 2y,\ 3y-3x)$$
$$= (3(0)+2(2y),\ 4(3y-3x)-0-3(2y))$$
$$= (4y,\ 12y-12x,\ -6y)$$

Ex. $T_1 : P_n \to P_n$ & $T_2 : P_n \to P_n$ be the L.T. given by $T_1(p(x)) = p(x-1)$ & $T_2(p(x)) = p(x+1)$
Find $(T_1 \circ T_2)(p(x))$ & $(T_2 \circ T_1)(p(x))$.

$\Rightarrow$ $(T_1 \circ T_2)(p(x)) = T_1[T_2(p(x))$
$$= T_1(p(x+1))$$
$$= p(x+1-1)$$
$$= p(x)$$

$(T_2 \circ T_1)(p(x)) = T_2(T_1(p(x))$
$$= T_2(p(x-1))$$
$$= p(x-1+1) = p(x).$$

Ex. $T_1 : P_2 \to P_2$, $T_2 : P_2 \to P_3$ are defined by
$T_1(p(x)) = p(x+1)$, $T_2(p(x)) = x\,p(x)$
Find $(T_2 \circ T_1)(a_0 + a_1 x + a_2 x^2)$.

$\Rightarrow$ $(T_2 \circ T_1)(a_0 + a_1 x + a_2 x^2)$
$$= T_2(T_1(a_0 + a_1 x + a_2 x^2))$$ put $x = x+1$
$$= T_2[a_0 + a_1(x+1) + a_2(x+1)^2]$$
$$= x[a_0 + a_1(x+1) + a_2(x+1)^2].$$

* Inverse of Linear Transformation -

Let $T_1 : V \to W$ be linear transformatio then there exist a linear transformation $T_2 : W \to V$ such that $T_2 \circ T_1 = I_V$, I_V is identity in V. then T_2 is called left inverse of T_1. Also if $T_1 \circ T_2 = I_W$, I_W is identity in W then T_2 is called right inverse of T_1.

If both left as well as right inverses exists then T_2 is called inverse of T_1.

Thm- A linear transformation $T : V \to W$ has an inverse if & only if it is bijective.

Ex. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be multiplication by A. Determine whether T has an inverse if so find $T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$ where

a) $A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$

$\Rightarrow \quad TX = AX$

$$T\begin{bmatrix} x \\ y \end{bmatrix} = T(x, y) = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5x+2y \\ 2x+y \end{bmatrix}$$

$$\therefore T(x, y) = (5x+2y, 2x+y)$$

TST T is bijective (1-1 & onto)

① For 1-1 we show that $\ker(T) = \{\bar{0}\}$

Consider

$$(x, y) \in \ker(T)$$

$$\therefore T(x, y) = \bar{0}$$

$$(5x+2y, 2x+y) = (0, 0)$$

$$5x + 2y = 0 \quad \text{—①}$$

$$2x + y = 0 \quad \text{—②}$$

$$5x+2y=0$$
$$4x+2y=0$$
$$\underline{-\quad -\quad -}$$
$$x=0$$

① − 2②

$x=0$

from ① $5(0)+2y=0$

$\Rightarrow 2y=0 \Rightarrow y=0$

$\therefore (x,y)=(0,0)$

$\therefore \ker(T)=\{\bar{0}\}$

$\therefore T$ is one-one.

① $T(x,y)=\bar{0}$

② $T(x,y)=(x_1,x_2)$

Find x,y

② onto

For $(x_1,x_2)\in\mathbb{R}^2$ find (x,y) s.t.

$$T(x,y)=(x_1,x_2) \quad \text{— } (*)$$

$$(5x+2y,\ 2x+y)=(x_1,x_2)$$

$$5x+2y=x_1 \quad \text{— } ①$$
$$2x+y=x_2 \quad \text{— } ②$$

① − 2②

$$5x+2y=x_1$$
$$4x+2y=2x_2$$
$$\underline{-\quad -\quad -}$$
$$x=x_1-2x_2$$

put $x=x_1-2x_2$ in ②

$$2(x_1-2x_2)+y=x_2$$
$$y=x_2-2x_1+4x_2$$
$$y=-2x_1+5x_2$$

$\therefore T$ is onto

$\therefore T^{-1}$ is exist.

from (*)

$$T(x,y)=(x_1,x_2)$$
$$\Rightarrow (x,y)=T^{-1}(x_1,x_2)$$

$$T^{-1}(x_1, x_2) = (x, y) = (x_1 - 2x_2, -2x_1 + 5x_2)$$

or $$T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ -2x_1 + 5x_2 \end{bmatrix}$$

b) $$A = \begin{bmatrix} 6 & -3 \\ 4 & -2 \end{bmatrix}$$

or $|A| = -12 + 12 = 0$

A^{-1} is not exist

$\therefore T^{-1}$ is not exist.

$$TX = AX$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 6 & -3 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 6x - 3y \\ 4x - 2y \end{bmatrix}$$

$$\therefore T(x, y) = (6x - 3y, 4x - 2y)$$

① one-one

consider $T(x, y) = \bar{0}$

$$(6x - 3y, 4x - 2y) = (0, 0)$$

$$6x - 3y = 0$$

$$4x - 2y = 0$$

$$\left[\begin{array}{cc|c} 6 & -3 & 0 \\ 4 & -2 & 0 \end{array}\right]$$

$\frac{1}{6}R_1$

$$\sim \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 4 & -2 & 0 \end{array}\right]$$

$R_2 - 4R_1$

$$\begin{array}{ccc} x & y & = \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Take y as a free variable

$\therefore y = t, \quad t \in \mathbb{R}$

$\therefore x - \frac{1}{2}y = 0 \Rightarrow x = \frac{t}{2}$

$(x,y) = \left(\frac{t}{2}, t\right) , \quad t \in \mathbb{R}$

$\neq 0$

$\therefore \ker(T) \neq \{\bar{0}\}$

$\therefore T$ is not one-one.

$\therefore T$ is not bijective

$\therefore T^{-1}$ does not exist.

c] $A = \begin{bmatrix} 4 & 7 \\ -1 & 3 \end{bmatrix}$

$|A| = 12 + 7 = 19 \neq 0$

$\therefore A^{-1}$ is exist.

$\therefore T^{-1}$ exist.

$\Rightarrow \quad Tx = Ax$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 4 & 7 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4x+7y \\ -x+3y \end{bmatrix}$

① 1-1

$T(x,y) = \bar{0}$

$\Rightarrow x = 0, y = 0$

② onto

$T(x,y) = (x_1, x_2)$

$T(x,y) = (4x+7y, -x+3y)$

① one-one, Let $(x,y) \in \ker(T)$

consider $T(x,y) = \bar{0}$

$(4x+7y, -x+3y) = (0,0)$

$4x + 7y = 0 \quad$ — ①

$-x + 3y = 0 \quad$ — ②

① + 4 ②

$\cancel{4x} + 7y = 0$

$-\cancel{4x} + 12y = 0$

$19y = 0$

$\Rightarrow y = 0$

put $y = 0$ in ②

$-x + 0 = 0 \Rightarrow x = 0$

$\therefore (x,y) = (0,0)$

$\therefore \ker(T) = \{\bar{0}\}$

$\therefore T$ is one-one.

② onto –

consider $T(x,y) = (x_1, x_2)$

$$(4x+7y, -x+3y) = (x_1, x_2)$$

$$4x + 7y = x_1 \quad —①$$

$$-x + 3y = x_2 \quad —②$$

① + 4 ②

$$19y = x_1 + 4x_2$$

$$y = \frac{1}{19}(x_1 + 4x_2)$$

from ②

$$-x + \frac{3}{19}(x_1 + 4x_2) = x_2$$

$$x = \frac{3}{19}x_1 + \frac{12}{19}x_2 - x_2$$

$$= \frac{3}{19}x_1 - \frac{7}{19}x_2$$

$$x = \frac{1}{19}(3x_1 - 7x_2)$$

$\therefore T$ is onto, $\therefore T^{-1}$ is exist.

③ find inverse –

$$T(x,y) = (x_1, x_2)$$

$$\therefore (x,y) = T^{-1}(x_1, x_2)$$

$$T^{-1}(x_1, x_2) = (x, y)$$

$$= \left(\frac{1}{19}(3x_1 - 7x_2), \frac{1}{19}(x_1 + 4x_2)\right)$$

$$\therefore T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{19}(3x_1 - 7x_2) \\ \frac{1}{19}(x_1 + 4x_2) \end{bmatrix}$$

Ex. Determine whether $T(x,y) = (x+2y, x-2y)$

is invertible or not.

$\Rightarrow$ (1) one-one -

Let $(x,y) \in \ker(T)$

$\therefore T(x,y) = \bar{0}$

$(x+2y, x-2y) = (0,0)$

$x+2y=0$ —— (1)

$x-2y=0$ —— (2)

(1) + (2)

$2x=0$

$\Rightarrow x=0$

put $x=0$ in (1)

$0+2y=0$

$\Rightarrow y=0$

$(x,y)=(0,0)$

$\therefore \ker(T) = \{\bar{0}\}$

$\therefore T$ is one-one.

(2) onto -

consider $T(x,y) = (x_1, x_2)$

$(x+2y, x-2y) = (x_1, x_2)$

$x+2y = x_1$ —— (1)

$x-2y = x_2$ —— (2)

(1) + (2)

$2x = x_1 + x_2$

$x = \frac{x_1}{2} + \frac{x_2}{2}$

put $x = \frac{x_1}{2} + \frac{x_2}{2}$ in (1)

$\frac{x_1}{2} + \frac{x_2}{2} + 2y = x_1$

$2y = x_1 - \frac{x_1}{2} - \frac{x_2}{2} = \frac{x_1}{2} - \frac{x_2}{2}$

$y = \frac{x_1}{4} - \frac{x_2}{4}$

$\therefore T$ is onto, $\therefore T^{-1}$ is exist.

Inverse –

$$T(x,y) = (x_1, x_2)$$
$$(x,y) = T^{-1}(x_1, x_2)$$
$$T^{-1}(x_1,x_2) = \left(\frac{x_1}{2} + \frac{x_2}{2},\ \frac{x_1}{4} - \frac{x_2}{4}\right)$$

Ex. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be multiplication by A. Determine whether T has an inverse. if so find $T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)$ where

a) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$\Rightarrow$ $Tx = Ax$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+z \\ y+z \\ x+y \end{bmatrix}$$

$\therefore T(x,y,z) = (x+z,\ y+z,\ x+y)$

① one-one

$(x,y,z) \in \ker(T)$

$\therefore T(x,y,z) = \bar{0}$

$(x+z,\ y+z,\ x+y) = (0,0,0)$

$x+z = 0$ — ①

$y+z = 0$ — ②

$x+y = 0$ — ③

① – ②

$x-y = 0$ — ④

③ + ④

$2x = 0 \Rightarrow x = 0$

from ③

$y=0$

from ①

$z=0$

$(x,y,z)=(0,0,0)$

$\therefore \ker(T)=\{\bar{0}\}$

② Onto

consider $T(x,y,z)=(x_1,x_2,x_3)$

$(x+z,\ y+z,\ x+y)=(x_1,x_2,x_3)$

$x+z=x_1$ — ①

$y+z=x_2$ — ②

$x+y=x_3$ — ③

① − ②

$x-y=x_1-x_2$ — ④

③ + ④

$2x=x_1-x_2+x_3$

$\therefore x=\frac{x_1}{2}-\frac{x_2}{2}+\frac{x_3}{2}$

From ③

$\frac{x_1}{2}-\frac{x_2}{2}+\frac{x_3}{2}+y=x_3$

$y=x_3-\frac{x_1}{2}+\frac{x_2}{2}-\frac{x_3}{2}$

$=\frac{x_3}{2}-\frac{x_1}{2}+\frac{x_2}{2}$

from ①

$\frac{x_1}{2}-\frac{x_2}{2}+\frac{x_3}{2}+z=x_1$

$z=\frac{x_1}{2}+\frac{x_2}{2}-\frac{x_3}{2}$

$\therefore$ T is onto

$\therefore$ T is bijective

$\therefore$ T^{-1} exist.

Inverse -

consider $T(x,y,z) = (x_1, x_2, x_3)$

$\therefore (x,y,z) = T^{-1}(x_1, x_2, x_3)$

$\therefore T^{-1}(x_1,x_2,x_3) = \left(\frac{x_1}{2} - \frac{x_2}{2} + \frac{x_3}{2}, \frac{x_3}{2} - \frac{x_1}{2} + \frac{x_2}{2}, \frac{x_1}{2} + \frac{x_2}{2} - \frac{x_3}{2} \right)$

$$T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} \frac{x_1}{2} - \frac{x_2}{2} + \frac{x_3}{2} \\ \frac{x_3}{2} - \frac{x_1}{2} + \frac{x_2}{2} \\ \frac{x_1}{2} + \frac{x_2}{2} - \frac{x_3}{2} \end{bmatrix}.$$

Ex. $A = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

$\Rightarrow$ $TX = AX$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+5y+2z \\ x+2y+z \\ -x+y \end{bmatrix}$$

$T(x,y,z) = (x+5y+2z, x+2y+z, -x+y)$

(a) 1-1.

consider $(x,y,z) \in \ker(T)$

$\therefore T(x,y,z) = \bar{0}$

$(x+5y+2z, x+2y+z, -x+y) = (0,0,0)$

$x+5y+2z = 0$

$x+2y+z = 0$

$-x+y = 0$

$$\left[\begin{array}{ccc|c} 1 & 5 & 2 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right]$$

$R_2 - R_1,\ R_3 + R_1$

$$\begin{bmatrix} 1 & 5 & 2 & : & 0 \\ 0 & -3 & -1 & : & 0 \\ 0 & 6 & 2 & : & 0 \end{bmatrix}$$

$-\frac{1}{3} R_2$

$$\sim \begin{bmatrix} 1 & 5 & 2 & : & 0 \\ 0 & 1 & \frac{1}{3} & : & 0 \\ 0 & 6 & 2 & : & 0 \end{bmatrix}$$

$R_3 - 6R_2$

$$\sim \begin{array}{c} \begin{matrix} x & y & z & = & \end{matrix} \\ \begin{bmatrix} 1 & 5 & 2 & : & 0 \\ 0 & 1 & \frac{1}{3} & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \end{array}$$

Take z as a free variable

$z = t,\ t \in \mathbb{R}$

$x + 5y + 2z = 0$ —— (1)

$y + \frac{1}{3} z = 0$

$y = -\frac{t}{3}$

from (1)

$x - \frac{5}{3} t + 2t = 0$

$x + \frac{1}{3} t = 0$

$x = -\frac{1}{3} t$

$\therefore\ (x, y, z) = \left(-\frac{t}{3}, -\frac{t}{3}, t\right) \neq 0,\ t \in \mathbb{R}$

$\therefore\ \ker(T) \neq \{\bar{0}\}$

$\therefore$ T is not one-one

$\therefore$ T^{-1} does not exist.

Reference: A textbook of S.Y.B.Sc., Linear Algebra by Golden Series, Nirali Publication.