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Chapter 4: Linear Transformation

Topic- Matrix of a linear transformation

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* Matrix of Linear Transformation -

Let V & W be the vector spaces over a field F with the dimensions n and m resp. Let $B = \{\bar{v}_1, \bar{v}_2, \ldots \bar{v}_n\}$ and $B' = \{\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_m\}$ be the bases of V & W resp. Let $T: V \to W$ be a linear transformation, then for each $i = 1, 2, \ldots, n$, $T(\bar{v}_i) \in W$, hence is a linear combination of basis vectors of W. Therefore

$$T(\bar{v}_1) = a_{11}\bar{w}_1 + a_{21}\bar{w}_2 + \cdots + a_{m1}\bar{w}_m$$
$$T(\bar{v}_2) = a_{12}\bar{w}_1 + a_{22}\bar{w}_2 + \cdots + a_{m2}\bar{w}_m$$
$$\vdots$$
$$T(\bar{v}_n) = a_{1n}\bar{w}_1 + a_{2n}\bar{w}_2 + \cdots + a_{mn}w_m$$

where $a_{ij} \in \mathbb{R}$ for $i = 1, 2, \ldots, n$ & $j = 1, 2, \ldots m$.
Then matrix of T with respect to the bases B & B' is denoted by $[T]_B^{B'}$ & is given by.

$$[T]_B^{B'} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Thm Let $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. If $B = \{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n\}$ is the standard basis for $\mathbb{R}^n$, then T is multiplication by the matrix

$$A = [T(\bar{e}_1) \mid T(\bar{e}_2) \mid \cdots \mid T(\bar{e}_n)]$$

where $T(\bar{e}_1), T(\bar{e}_2), \ldots T(\bar{e}_n)$ are images of $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n$ considered as column vectors

Ex Find the standard matrix for the

Linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (3x-2y+z,\ 2x-3y,\ y-4z)$ and use it to find $T(2,-1,-1)$.

$\Rightarrow$ standard basis for $\mathbb{R}^3$ is $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$
where $\bar{e}_1 = (1,0,0)$, $\bar{e}_2 = (0,1,0)$, $\bar{e}_3 = (0,0,1)$

$$T(\bar{e}_1) = T(1,0,0) = (3,2,0)$$
$$T(\bar{e}_2) = T(0,1,0) = (-2,-3,1)$$
$$T(\bar{e}_3) = T(0,0,1) = (1,0,-4)$$

standard matrix is

$$[T] = [\,T(\bar{e}_1) \mid T(\bar{e}_2) \mid T(\bar{e}_3)\,]$$

$$= \begin{bmatrix} 3 & -2 & 1 \\ 2 & -3 & 0 \\ 0 & 1 & -4 \end{bmatrix}$$

To find $T(2,-1,-1)$ (by using matrix)

$TX = AX$

$$T\left(\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 3 & -2 & 1 \\ 2 & -3 & 0 \\ 0 & 1 & -4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6+2-1 \\ 4+3+0 \\ 0-1+4 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 3 \end{bmatrix}$$

$$T(2,-1,-1) = (7,7,3).$$

Ex. Find standard matrix for the L.T. $T: \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T(x,y,z) = (3x-4y+z,\ x+y-z,\ y+z,\ x+2y+3z)$$

$\Rightarrow$ standard basis for $\mathbb{R}^3$

$$B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$$

where $\bar{e}_1 = (1,0,0)$, $\bar{e}_2 = (0,1,0)$, $\bar{e}_3 = (0,0,1)$

$T(\bar{e}_1) = T(1,0,0) = (3,1,0,1)$,

$T(\bar{e}_2) = T(0,1,0) = (-4,1,1,2)$

$T(\bar{e}_3) = T(0,0,1) = (1,-1,1,3)$

$$\therefore [T] = \begin{bmatrix} 3 & -4 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

Ex. Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_1(x,y) = (x-2y, 2x+3y)$ & $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_2(x,y) = (y,0)$. compute the standard matrix of $T_2 \circ T_1$.

$\Rightarrow$ $\underset{②}{T_2} \circ \underset{①}{T_1} : \mathbb{R}^2 \to \mathbb{R}^2$

$(T_2 \circ T_1)(x,y) = T_2(T_1(x,y))$

$= T_2(x-2y, 2x+3y)$

$(T_2 \circ T_1)(x,y) = (2x+3y, 0)$

standard basis for $\mathbb{R}^2$ is $\{\bar{e}_1, \bar{e}_2\}$

$\bar{e}_1 = (1,0)$, $\bar{e}_2 = (0,1)$

$\therefore (T_2 \circ T_1)(\bar{e}_1) = (T_2 \circ T_1)(1,0) = (2,0)$

$(T_2 \circ T_1)(\bar{e}_2) = (T_2 \circ T_1)(0,1) = (3,0)$

$\therefore$ standard matrix of $(T_2 \circ T_1)$ is

$$[T_2 \circ T_1] = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$$

Ex. Find the standard matrix for the L.T.

$T : \mathbb{R}^3 \to \mathbb{R}^2$ defined $T(x,y,z) = (2x+y, 3y-z)$
and use it to find $T(0,1,-1)$.

$\Rightarrow$ standard basis for $\mathbb{R}^3$ is $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$

$T(\bar{e}_1) = T(1,0,0) = (2,0)$
$T(\bar{e}_2) = T(0,1,0) = (1,3)$
$T(\bar{e}_3) = T(0,0,1) = (0,-1)$

$$[T] = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & -1 \end{bmatrix}$$

$$Tx = Ax$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1+0 \\ 0+3+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$\therefore$ $T(0,1,-1) = (1,4)$.

Ex Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a L.T. defined by
$T(x,y,z) = (3x+y+z, x-3y-z)$
Find the matrix of T w.r.t. the bases
$B = \{(1,1,1), (-1,0,1), (0,0,1)\}$ and
$B' = \{(1,2), (-1,1)\}$ of $\mathbb{R}^3$ & $\mathbb{R}^2$ respectively.

$\Rightarrow$ ① $T(1,1,1) = (5,-3)$

$(5,-3) = k_1(1,2) + k_2(-1,1)$
$= (k_1, 2k_1) + (-k_2, k_2)$
$= (k_1 - k_2, 2k_1 + k_2)$

$\therefore$ $k_1 - k_2 = 5$ —①

$2k_1 + k_2 = -3$ —②

① + ②

$3k_1 = 2$

$k_1 = \frac{2}{3}$

From ① $\quad \frac{2}{3} - k_2 = 5$

$-k_2 = 5 - \frac{2}{3} = \frac{13}{3}$

$k_2 = -\frac{13}{3}$

1st col^n is $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -13/3 \end{bmatrix}$

② $T(-1,0,1) = (-2,-2)$

$(-2,-2) = k_1(1,2) + k_2(-1,1) = (k_1 - k_2, 2k_1 + k_2)$

$\therefore k_1 - k_2 = -2 \quad$ —①

$2k_1 + k_2 = -2 \quad$ —②

① + ②

$3k_1 = -4$

$k_1 = -\frac{4}{3}$

from ①

$-\frac{4}{3} - k_2 = -2$

$-k_2 = -2 + \frac{4}{3} = -\frac{2}{3}$

$k_2 = \frac{2}{3}$

2nd column = $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} \\ \frac{2}{3} \end{bmatrix}$

③ $T(0,0,1) = (1,-1)$

$(1,-1) = k_1(1,2) + k_2(-1,1) = (k_1 - k_2, 2k_1 + k_2)$

$k_1 - k_2 = 1 \quad$ —①

$2k_1 + k_2 = -1 \quad$ —②

$3k_1 = 0$

$\Rightarrow k_1 = 0$

from ① $\quad 0 - k_2 = 1 \Rightarrow k_2 = -1$

3rd column is $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$$\therefore [T]_B^{B'} = \begin{bmatrix} 2/3 & -4/3 & 0 \\ -13/3 & 2/3 & -1 \end{bmatrix}$$

Ex. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation defined by

$$T(x_1, x_2) = (x_2, -5x_1 + 13x_2, -7x_1 + 16x_2)$$

Find the matrix $[T]_B^{B'}$ where $B = \{(3,1), (5,2)\}$ & $B' = \{(1,0,-1), (-1,2,2), (0,1,2)\}$.

$\Rightarrow$ $T(3,1) = (1, -15+13, -21+16) = (1, -2, -5)$

$$(1,-2,-5) = k_1(1,0,-1) + k_2(-1,2,2) + k_3(0,1,2)$$

$$= (k_1, 0, -k_1) + (-k_2, 2k_2, 2k_2) + (0, k_3, 2k_3)$$

$$= (k_1 - k_2, 2k_2 + k_3, -k_1 + 2k_2 + 2k_3)$$

$k_1 - k_2 = 1$ — (1)

$2k_2 + k_3 = -2$ — (2)

$-k_1 + 2k_2 + 2k_3 = -5$ — (3)

(1) + (3)

$k_2 + 2k_3 = -4$ — (4)

(2) − 2(4)

$$\begin{array}{r} 2k_2 + k_3 = -2 \\ \underset{-}{2k_2} \underset{-}{+} 4k_3 = \underset{+}{-}8 \\ \hline -3k_3 = 6 \end{array}$$

$k_3 = -2$

put $k_3 = -2$ in (4)

$k_2 - 4 = -4 \Rightarrow k_2 = 0$

from (1)

$$k_1 - 0 = 1 \Rightarrow k_1 = 1$$

$$\therefore 1^{st} \text{ col}^n = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

(2) $T(5,2) = (2,1,-3)$

$$(2,1,-3) = k_1(1,0,-1) + k_2(-1,2,2) + k_3(0,1,2)$$

$$= (k_1 - k_2,\ 2k_2 + k_3,\ -k_1 + 2k_2 + 2k_3)$$

$$\therefore k_1 - k_2 = 2 \quad \text{— (1)}$$

$$2k_2 + k_3 = 1 \quad \text{— (2)}$$

$$-k_1 + 2k_2 + 2k_3 = -3 \quad \text{— (3)}$$

(1) + (3)

$$k_2 + 2k_3 = -1 \quad \text{— (4)}$$

(2) − 2(4)

$$2k_2 + k_3 = 1$$

$$2k_2 + 4k_3 = -2$$

$$- \quad - \quad\quad +$$

$$-3k_3 = 3$$

$$k_3 = -1$$

put $k_3 = -1$ in (4)

$$k_2 - 2 = -1$$

$$\Rightarrow k_2 = 1$$

put $k_2 = 1$ in (1)

$$k_1 - 1 = 2$$

$$k_1 = 3$$

$$\therefore 2^{nd} \text{ column} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

$$\therefore [T]_B^{B'} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$

Ex. Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a L.T. defined by

$$T(x,y,z) = (3x+2y-4z,\ x-5y+2z)$$

Find a matrix of T w.r.t. basis B & B', where $B = \{(1,1,1),(1,1,0),(1,0,0)\}$ & $B' = \{(1,3),(2,5)\}$.

$\Rightarrow$ ① $T(1,1,1) = (1,-2)$

$(1,-2) = k_1(1,3) + k_2(2,5)$

$= (k_1+2k_2,\ 3k_1+5k_2)$

$k_1+2k_2 = 1$ —①

$3k_1+5k_2 = -2$ —②

3① − ②

$3k_1+6k_2 = 3$

$3k_1+5k_2 = -2$

$\underline{-\quad -\quad +}$

$k_2 = 5$

from ①

$k_1+10=1 \Rightarrow k_1 = -9$

$\therefore$ 1st column $= \begin{bmatrix} -9 \\ 5 \end{bmatrix}$

② $T(1,1,0) = (5,-4)$

$\therefore (5,-4) = k_1(1,3) + k_2(2,5) = (k_1+2k_2, 3k_1+5k_2)$

$\therefore k_1+2k_2 = 5$ —①

$3k_1+5k_2 = -4$ —②

3① − ②

$\cancel{3k_1}+6k_2 = 15$

$-\cancel{3k_1} \mp 5k_2 = \mp 4$

$k_2 = 19$

from ① $k_1+38=5$

$\Rightarrow k_1 = -33$

$$2^{nd} \text{ column} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -33 \\ 19 \end{bmatrix}$$

③ $T(1,0,0) = (3,1)$

$(3,1) = k_1(1,3) + k_2(2,5) = (k_1 + 2k_2, 3k_1 + 5k_2)$

$\therefore k_1 + 2k_2 = 3 \quad \text{—①}$

$3k_1 + 5k_2 = 1 \quad \text{—②}$

$3① - ②$

$k_2 = 8$

from ① $\quad k_1 + 16 = 3$

$\Rightarrow k_1 = -13$

$$3^{rd} \text{ column} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -13 \\ 8 \end{bmatrix}$$

$$\therefore [T]_{B}^{B'} = \begin{bmatrix} -9 & -17 & -13 \\ 5 & 11 & 8 \end{bmatrix}$$

Ex. Let $D : P_2 \to P_1$ be L.T. given by $D(p) = \frac{dp}{dx}$. Find the matrix of D relative to the standard bases $B_1 = \{1, x, x^2\}$, and $B_2 = \{1, x\}$ of P_2 & P_1 resp.

$\Rightarrow$

① $D(1) = \frac{d}{dx}(1) = 0$

$0 = k_1(1) + k_2(x) = k_1 + k_2 x$

$k_1 = 0, \; k_2 = 0$

$$\therefore 1^{st} \text{ col}^n = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

② $D(x) = \frac{d}{dx}(x) = 1$

$$1 = k_1(1) + k_2(x) = k_1 + k_2 x$$

$$k_1 = 1 \ \& \ k_2 = 0$$

$$2^{nd} \text{ column} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

③ $$D(x^2) = \frac{d}{dx}(x^2) = 2x$$

$$2x = k_1(1) + k_2(x) = k_1 + k_2 x$$

$$k_1 = 0 \ \& \ k_2 = 2$$

$$3^{rd} \text{ col}^n = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\therefore [T]_{B_1}^{B_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Ex. Let $V = M_{2\times 2}(\mathbb{R})$ be the vector space of all 2×2 matrices with real numbers. Let T be the operator on V defined by $T(X) = AX$, where $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Find the matrix of T w.r.t. the basis

$$B = \left\{ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \right.$$

$$\left. A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ for } V.$$

$\Rightarrow$ $$TX = AX$$

① $$T(A_1) = AA_1$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 0+0 \\ 1+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = k_1 A_1 + k_2 A_2 + k_3 A_3 + k_4 A_4$$

$$= k_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + k_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}$$

$k_1 = 1, \; k_2 = 0, \; k_3 = 1, \; k_4 = 0$

1^{st} col^n $= \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

(2) $T(A_2) = AA_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1+0 \\ 0 & 1+0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = k_1 A_1 + k_2 A_2 + k_3 A_3 + k_4 A_4 = \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}$$

$k_1 = 0, \; k_2 = 1, \; k_3 = 0, \; k_4 = 1$

$\therefore$ 2^{nd} col^n $= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

(3) $T(A_3) = AA_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}$$

$\therefore$ $k_1 = 1, \; k_2 = 0, \; k_3 = 1, \; k_4 = 0$

$\therefore$ 3^{rd} col^n $= \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

④ $T(A_4) = AA_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

$\quad$ lly

$\therefore\ k_1 = 0,\ k_2 = 1,\ k_3 = 0,\ k_4 = 1$

$\therefore$ 4^{th} coln $= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

$$\therefore [T]_B^B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

* Basis Matrix Transformations in $\mathbb{R}^2$ & $\mathbb{R}^3$.

* Reflection operators –

The operators on $\mathbb{R}^2$ & $\mathbb{R}^3$ which maps each point into its symmetric image about a fixed line or a fixed plane that contains the origin are called as reflection operators.

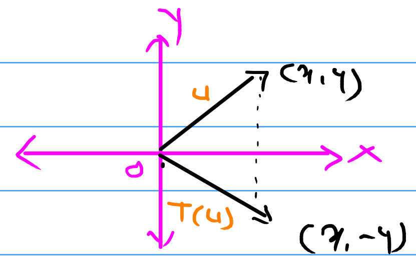
1] Reflection about x-axis

$$T(x, y) = (x, -y)$$

standard matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

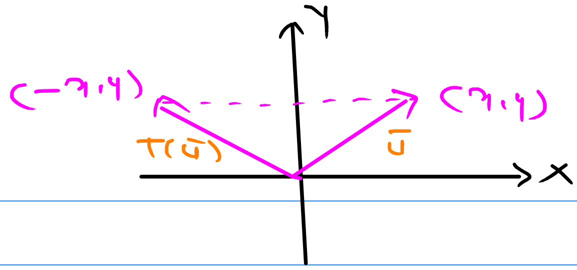
Geometrical image



2] Reflection about Y-axis

$$T(x, y) = (-x, y)$$

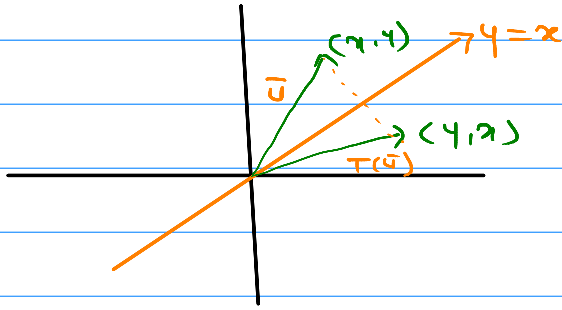
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



3] Reflection about $y = x$

$$T(x,y) = (y,x)$$

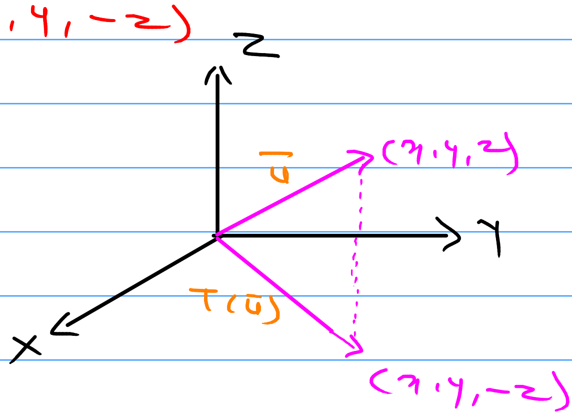
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



iv) Reflection about XY plane

$$T(x,y,z) = (x,y,-z)$$

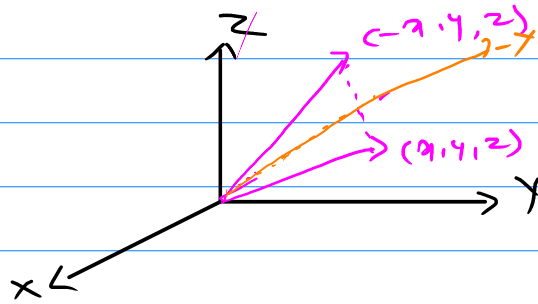
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$



v) Reflection about YZ plane -

$$T(x,y,z) = (-x,y,z)$$

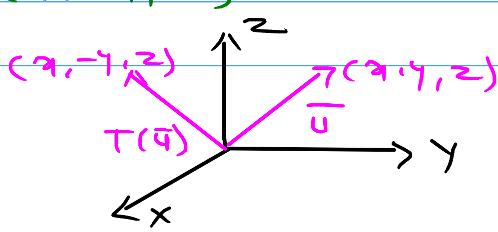
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



vi) Reflection about XZ plane

$$T(x,y,z) = (x,-y,z)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



	Reflection	Projection.
x-axis	$T(x,y) = (x,-y)$	$T(x,y) = (x,0)$
y-axis	$T(x,y) = (-x,y)$	$T(x,y) = (0,y)$
$y=x$ line	$T(x,y) = (y,x)$	
xy-plane	$T(x,y,z) = (x,y,-z)$	$T(x,y,z) = (x,y,0)$
yz-plane	$T(x,y,z) = (-x,y,z)$	$T(x,y,z) = (0,y,z)$
xz-plane	$T(x,y,z) = (x,-y,z)$	$T(x,y,z) = (x,0,z)$

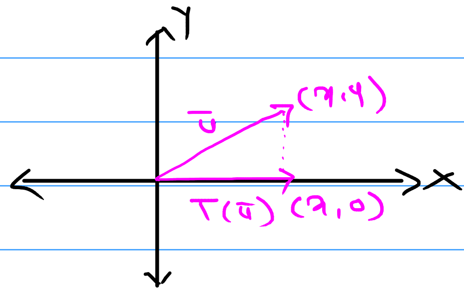
* Projection Operators -

Matrix operators on $\mathbb{R}^2$ and $\mathbb{R}^3$ that map each point into its orthogonal projection onto a fixed line or plane through the origin are called projection operators.

1) Orthogonal projection onto x-axis -

$$T(x,y) = (x,0)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

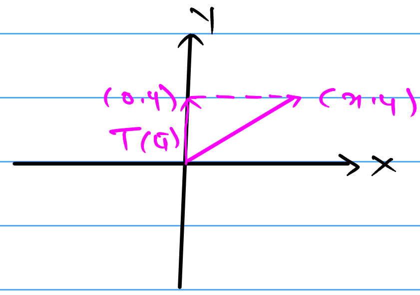


2] Orthogonal projection onto y-axis -

$$T(x,y) = (0,y)$$

Matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$



3] Orthogonal projection onto the xy plane -

$$T(x,y,z) = (x,y,0)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

4] Orthogonal projection onto the xz plane

$$T(x,y,z) = (x,0,z)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5] Orthogonal projection onto the yz plane

$$T(x,y,z) = (0,y,z)$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

* Rotation Operators -

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

$$T(\bar{e}_1) = T(1,0) = (\cos\theta, \sin\theta)$$

$$T(\bar{e}_2) = T(0,1) = (-\sin\theta, \cos\theta)$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Ex. Use matrix multiplication to find the reflection of $(-1,2)$ about the

a) x-axis b) y-axis c) line $y=x$.

$\Rightarrow$ a) The standard matrix for reflection about x-axis is

$T(x,y)=(x,-y)$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\therefore$ Reflection of $(-1,2)$ about the x-axis is

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1+0 \\ 0-2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

b) Reflection about y axis

$T(x,y)=(-x,y)$

$\therefore$ standard matrix is

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ Reflection of $(-1,2)$ about the y-axis is

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+0 \\ 0+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

c) Reflection about $y=x$ line

$T(x,y)=(y,x)$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\therefore$ Reflection of $(-1,2)$ about the line $y=x$ is

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0+2 \\ -1+0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Ex. Use matrix multiplication to find the reflection of (2,−5,3) about the

a) xy plane b) xz plane c) yz plane.

⇒ a) xy plane

$$T(x,y,z) = (x,y,-z)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

∴ Reflection of (2,−5,3) about xy plane is

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}$$

b) Reflection about xz plane -

$$T(x,y,z) = (x,-y,z)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

∴ Reflection of (2,−5,3) about xz plane is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$$

c) yz plane

$$T(x,y,z) = (-x,y,z)$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

∴ Reflection of (2,−5,3) about yz plane is

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ -5 \\ 3 \end{bmatrix}$$

Ex. Use matrix multiplication to find orthogonal projection of (a,b,c) onto the
a) xy plane b) xz-plane c) yz plane.

$\Rightarrow$ Orthogonal projection onto xy plane

$$T(x,y,z) = (x,y,0)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The orthogonal projection of (a,b,c) onto xy plane is

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$$

2] xz-plane

$$T(x,y,z) = (x,0,z)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The orthogonal projection of (a,b,c) onto xz plane is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ c \end{bmatrix}$$

c) YZ plane -

$$T(x,y,z) = (0,y,z)$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ c \end{bmatrix}$$

Ex. Use matrix multiplication to find the image of the vector $(3,-4)$ when it is rotated about the origin through an angle of a) $\theta = 30°$, b) $\theta = -60°$ c) $\theta = 45°$ d) $\theta = 90°$.

$\Rightarrow$

Rotation $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

a) $\theta = 30°$

$$\begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Image of the vector $(3,-4)$ is

$$= \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 3\frac{\sqrt{3}}{2} + 2 \\ \frac{3}{2} - 2\sqrt{3} \end{bmatrix}$$

$\cos(-\theta) = \cos\theta$

$\sin(-\theta) = -\sin\theta$

b) $\theta = -60°$

$\therefore$ std. Matrix $= \begin{bmatrix} \cos(-60°) & -\sin(-60°) \\ \sin(-60°) & \cos(-60°) \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

Image of the vector $(3,-4)$ is

$$= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} - 2\sqrt{3} \\ -\frac{3\sqrt{3}}{2} - 2 \end{bmatrix}$$

c) $\theta = 45^\circ$

standard matrix $= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

Image of the vector $(3,-4)$ is

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{\sqrt{2}} + \frac{4}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} - \frac{4}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{7}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

* **Linear Isomorphism** –

A linear transformation $T: V \to W$ that both one-one and onto is said to be an linear isomorphism and V is said to be isomorphic to W.

Thm Every real n-dimensional vector space is isomorphic to $\mathbb{R}^n$.

Proof - Let V be any n-dimensional vector space

$\therefore$ Basis of V contain n vectors.

Let $S = \{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n\}$ be basis for V.

Let $\bar{u} \in V$

$\therefore \quad \bar{u} = k_1\bar{v}_1 + k_2\bar{v}_2 + \dots + k_n\bar{v}_n$

Define $T : V \longrightarrow \mathbb{R}^n$

$T(\bar{u}) = (k_1, k_2, \dots, k_n)$

① TST T is L.T.

Let $\bar{u}, \bar{v} \in V$

$\therefore \quad \bar{u} = k_1\bar{v}_1 + k_2\bar{v}_2 + \dots + k_n\bar{v}_n$

$\bar{v} = c_1\bar{v}_1 + c_2\bar{v}_2 + \dots + c_n\bar{v}_n$

$\bar{u} + \bar{v} = (k_1+c_1)\bar{v}_1 + (k_2+c_2)\bar{v}_2 + (k_n+c_n)\bar{v}_n$

$T(\bar{u}+\bar{v}) = (k_1+c_1, k_2+c_2, \dots, k_n+c_n) \longrightarrow ①$

$T(\bar{u}) = (k_1, k_2, \dots, k_n)$

$T(\bar{v}) = (c_1, c_2, \dots, c_n)$

$T(\bar{u}) + T(\bar{v}) = (k_1+c_1, k_2+c_2, \dots, k_n+c_n) \longrightarrow ②$

$\therefore \quad T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v})$

$\alpha\bar{u} = \alpha k_1\bar{v}_1 + \alpha k_2\bar{v}_2 + \dots + \alpha k_n\bar{v}_n$

$T(\alpha\bar{u}) = (\alpha k_1, \alpha k_2, \dots, \alpha k_n)$

$= \alpha(k_1, k_2, \dots, k_n)$

$T(\alpha\bar{u}) = \alpha T(\bar{u})$

$\therefore$ T is linear transformation

ⓑ TST T is one-one

if $\bar{u} \neq \bar{v}$

$\therefore \quad k_i \neq c_i$ for some i

$(k_1, k_2, \dots, k_n) \neq (c_1, c_2, \dots, c_n)$

$\therefore \quad T(\bar{u}) \neq T(\bar{v})$

$\therefore T$ is one - one.

ⓒ Let $\bar{\omega} \in \mathbb{R}^n$

& $\bar{\omega} = (k_1, k_2, \ldots, k_n)$

Then for $\bar{u} \in V$ s.t.

$$\bar{u} = k_1\bar{v}_1 + k_2\bar{v}_2 + \cdots + k_n\bar{v}_n$$

$$T(\bar{u}) = (k_1, k_2, \ldots, k_n) = \bar{\omega}$$

$\therefore T$ onto.

$\therefore T$ isomorphism

$\therefore V$ & $\mathbb{R}^n$ are isomorphic.

<u>Thm</u> Let V be a finite dimensional vector space and $T: V \to V$ is linear transformation then following statements are equivalent.

i) T is an isomorphism

ii) $\ker(T) = \{\bar{0}\}$

iii) $\operatorname{Im}(T) = V$

$\Rightarrow$ i) $\Rightarrow$ ii)

Consider T is an isomorphism

TST $\ker(T) = \{\bar{0}\}$

Let $\bar{u} \in \ker(T)$

$\Rightarrow T(\bar{u}) = \bar{0}$ 1-1

since $T(\bar{0}) = \bar{0}$ $f(x) = f(y)$

$\therefore T(\bar{u}) = T(\bar{0})$ $x = y$

since T is an isomorphism

$\therefore T$ is one - one

$\bar{u} = \bar{0}$

$\therefore \ker(T) = \{\bar{0}\}$

ii) $\Rightarrow$ iii)

consider $\ker(T) = \{\bar{0}\}$

$\text{Basis } \ker(T) = \{\ \}$

$\text{nullity}(T) = 0$

$\therefore$ By dimension thm

$\text{rank}(T) + \text{nullity}(T) = \dim(V)$

$\dim R(T) + 0 = \dim V$

or $\dim Im(T) = \dim V$

$\therefore Im(T) = V$

Range = Image

iii) $\Rightarrow$ i)

consider $Im(T) = V$

$\therefore$ T is onto

TST T is isomorphism it is sufficient to show that T is one-one (T is L.T. given)

consider $T(\bar{u}) = T(\bar{v})$

$\therefore T(\bar{u}) - T(\bar{v}) = \bar{0}$

$\therefore T(\bar{u} - \bar{v}) = \bar{0}$

$\therefore \bar{u} - \bar{v} \in \ker(T)$

if $\bar{u} \neq \bar{v}$ then $\bar{u} - \bar{v} \neq \bar{0}$

$\therefore \ker(T)$ contain non zero vector

$\therefore \dim \ker(T) \geq 1$

i.e. $\text{nullity}(T) \geq 1$ — $(*)$

$\therefore \text{rank}(T) + \text{nullity}(T) = \dim V$ — (1)

since $Im(T) = V$

$\therefore \dim Im(T) = \dim V$

i.e. $\text{rank}(T) = \dim V$

from (1)

$\dim V + \text{nullity}(T) = \dim V$

$\therefore$ nullity$(T) = 0$

which is contradiction to *

$\therefore \quad \bar{u} = \bar{v}$

$\therefore$ T is one-one.

Hence T is an isomorphism.

Thm Let $T: V \to W$ be a linear transformation, if T is an isomorphism, then T maps linearly independent sets in V to linearly independent set in W.

Proof Let $S = \{\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_n\}$ be linearly independent in V.

TST $S' = \{T(\bar{v}_1), T(\bar{v}_2), \cdots, T(\bar{v}_n)\}$ is linearly independent.

consider

$$k_1 T(\bar{v}_1) + k_2 T(\bar{v}_2) + \cdots + k_n T(\bar{v}_n) = \bar{0}$$

$$\therefore T(k_1\bar{v}_1 + k_2\bar{v}_2 + \cdots + k_n\bar{v}_n) = \bar{0}$$

since $T(\bar{0}) = \bar{0}$

$$\therefore T(k_1\bar{v}_1 + k_2\bar{v}_2 + \cdots + k_n\bar{v}_n) = T(\bar{0})$$

since T is isomorphism

$\therefore$ T is one-one

$$k_1\bar{v}_1 + k_2\bar{v}_2 + \cdots + k_n\bar{v}_n = \bar{0}$$

since $\{\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_n\}$ is L.I.

$$\therefore k_1 = k_2 = \cdots = k_n = 0$$

$\therefore \{T(\bar{v}_1), T(\bar{v}_2), \cdots, T(\bar{v}_n)\}$ are L.I.

Ex Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be map defined by $T(x, y) = (x+y, x-y)$ show that T is isomorphism.

$\Rightarrow$ ① L.T.

Let $\bar{u} = (x, y)$, $\bar{v} = (x', y')$

$\therefore \bar{u} + \bar{v} = (x + x', y + y')$

$T(\bar{u} + \bar{v}) = T(x + x', y + y') = (x + x' + y + y', x + x' - y - y')$

$T(\bar{u}) = (x + y, x - y')$ & $T(\bar{v}) = (x' + y', x' - y')$

$T(\bar{u}) + T(\bar{v}) = (x + y + x' + y', x - y + x' - y')$

$\therefore T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v})$

$k\bar{u} = (kx, ky)$

consider $T(k\bar{u}) = T(kx, ky) = (kx + ky, kx - ky)$

$= k(x + y, x - y)$

$= k\,T(x, y)$

$= k\,T(\bar{u})$

$\therefore$ T is L.T.

(2) one-one

consider $T(x, y) = \bar{0}$

$(x + y, x - y) = (0, 0)$

$x + y = 0$ — (1)

$x - y = 0$ — (2)

(1) + (2)

$2x = 0 \Rightarrow x = 0$

from (1)

$y = 0$

$\therefore (x, y) = (0, 0)$

$\therefore \ker(T) = \{\bar{0}\}$

$\therefore$ T is one-one

(3) onto — For $(a, b) \in \mathbb{R}^2$

consider $T(x, y) = (a, b)$

$(x + y, x - y) = (a, b)$

$$x+y=a \quad \text{--- } ①$$
$$x-y=b \quad \text{--- } ②$$
① + ②
$$2x=a+b \Rightarrow x=\frac{a}{2}+\frac{b}{2}$$

From ①
$$\frac{a}{2}+\frac{b}{2}+y=a$$
$$y=\frac{a}{2}-\frac{b}{2}$$

$\therefore$ For $(a,b)\in\mathbb{R}^2$ $\exists$ $(x,y)=\left(\frac{a}{2}+\frac{b}{2}, \frac{a}{2}-\frac{b}{2}\right)\in\mathbb{R}^2$

s.t. $T(x,y)=(a,b)$

$\therefore$ T is onto

$\therefore$ T is an isomorphism.

Ex. Let $T:\mathbb{R}^3\to\mathbb{R}^3$ be linear transformation defined by $T(x,y,z)=(-x,-y,-z)$. S.T. T is linear isomorphism.

$\Rightarrow$ ① T is L.T. (given)

② TST T is one-one

consider $T(x,y,z)=\bar{0}$
$$(-x,-y,-z)=(0,0,0)$$
$$-x=0,\ -y=0,\ -z=0$$
$$\Rightarrow x=0,\ y=0,\ z=0$$
$\therefore (x,y,z)=(0,0,0)$

$\therefore \ker(T)=\{\bar{0}\}$

T is one-one.

③ T is onto -

For $(a,b,c)\in\mathbb{R}^3$

consider $T(x,y,z)=(a,b,c)$
$$(-x,-y,-z)=(a,b,c)$$
$$x=-a,\ y=-b,\ z=-c$$

$\therefore$ For $(a,b,c) \in \mathbb{R}^3$ $\exists$ $(x,y,z)=(-a,-b,-c)\in\mathbb{R}^3$
s.t. $T(x,y,z) = (a,b,c)$.
$\therefore$ T is onto.
$\therefore$ T is an isomorphism.

Ex. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined as
$T(x,y) = (x+2y, 3x-y)$. S.T. T is an linear isomorphism.

$\Rightarrow$ ① S.T. T is L.T.

$\bar{u} = (x,y)$, $\bar{v} = (x',y')$

$\therefore \bar{u}+\bar{v} = (x+x', y+y')$

$T(\bar{u}+\bar{v}) = T(x+x', y+y')$

$= (x+x'+2y+2y', 3x+3x'-y-y')$

$T(\bar{u}) = (x+2y, 3x-y)$, $T(\bar{v}) = (x'+2y', 3x'-y')$

$T(\bar{u})+T(\bar{v}) = (x+2y+x'+2y', 3x-y+3x'-y')$

$\therefore T(\bar{u}+\bar{v}) = T(\bar{u})+T(\bar{v})$

$k\bar{u} = (kx, ky)$

$T(k\bar{u}) = T(kx, ky) = (kx+2ky, 3kx-ky)$

$= k(x+2y, 3x-y)$

$= kT(\bar{u})$

$\therefore$ T is L.T.

② TST T is one-one

consider $T(x,y) = \bar{0}$

$(x+2y, 3x-y) = (0,0)$

$x+2y = 0$ — ①

$3x-y = 0$ — ②

① + 2②

$7x = 0 \Rightarrow x = 0$

put $x=0$ in ①

$y=0$

$\therefore (x,y)=(0,0)$

$\ker(T)=\{\bar{0}\}$

$\therefore T$ is one-one.

③ TST T is onto

For $(a,b)\in\mathbb{R}^2$

consider $T(x,y)=(a,b)$

$(x+2y, 3x-y)=(a,b)$

$x+2y=a$ — ①

$3x-y=b$ — ②

① + 2②

$7x=a+2b$

$x=\frac{a}{7}+\frac{2b}{7}$

From ①

$\frac{a}{7}+\frac{2}{7}b+2y=a$

$2y=a-\frac{a}{7}-\frac{2}{7}b$

$2y=\frac{6}{7}a-\frac{2}{7}b$

$y=\frac{3a}{7}-\frac{b}{7}$

$\therefore$ For $(a,b)\in\mathbb{R}^2$ $\exists$ $(x,y)=\left(\frac{a}{7}+\frac{2}{7}b, \frac{3a}{7}-\frac{b}{7}\right)\in\mathbb{R}^2$

s.t. $T(x,y)=(a,b)$

$\therefore T$ is onto

$\therefore T$ is an linear isomorphism.

Reference: A textbook of S.Y.B.Sc., Linear Algebra by Golden Series, Nirali Publication.