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Chapter 4: Vector-Valued Functions

Topic- Integrals of Vector Functions, Arc Length along a Space Curve, Speed on a Smooth Curve, Unit Tangent Vector, Curvature of a Plane Curve, Circle of Curvature for Plane Curves, Curvature and Normal Vectors for a Space Curve.

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* Integrals of vector functions -

$\int x^n = \frac{x^{n+1}}{n+1}$

Ex. $\int_1^2 [(6-6t)\bar{i} + 3\sqrt{t}\,\bar{j} + \frac{4}{t^2}\bar{k}]\,dt$

$= [(6t - \cancel{6}^{3}\frac{t^2}{\cancel{2}_1})\bar{i} + \cancel{3}\frac{t^{3/2}}{\frac{3}{2}}\bar{j} + 4(\frac{-1}{t})\bar{k}]_{t=1}^{2}$

$\int \sqrt{t} = \int t^{1/2} = \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{t^{3/2}}{3/2}$

$= [(12-12)\bar{i} + 2(2)^{\frac{3}{2}}\bar{j} + 2\bar{k}]$

$- [(6-3)\bar{i} + 2\bar{j} + 4\bar{k}]$

$= -3\bar{i} + 4\sqrt{2}\,\bar{j} + 2\bar{k} - 2\bar{j} - 4\bar{k}$

$= -3\bar{i} + (4\sqrt{2}-2)\bar{j} - 2\bar{k}$

$\int \frac{1}{t^2} = \int t^{-2} = \frac{t^{-1}}{-1} = \frac{-1}{t}$

$2^{\frac{3}{2}} = (2^{\frac{1}{2}})^3 = (\sqrt{2})^3 = 2\sqrt{2}$

Ex. $\int_1^4 [\frac{1}{t}\bar{i} + \frac{1}{5-t}\bar{j} + \frac{1}{2t}\bar{k}]\,dt$

$\int \frac{1}{ax+b} = \log(ax+b)\frac{1}{a}$

$= [\log t\,\bar{i} + \log(5-t)\frac{1}{(-1)}\bar{j} + \frac{1}{2}\log t\,\bar{k}]_{t=1}^{4}$

$= [\log 4\,\bar{i} - \log(1)\bar{j} + \frac{1}{2}\log 4\,\bar{k}]$

$-[\log 1\,\bar{i} - \log(4)\bar{j} + \frac{1}{2}\log 1\,\bar{k}]$

$= \log 4\,\bar{i} - 0 + \frac{1}{2}\log 4\,\bar{k} + \log 4\,\bar{j}$

$= \log 4\,\bar{i} + \frac{1}{2}\log 4\,\bar{k} + \log 4\,\bar{j}$

$\frac{1}{2}\log 4 = \log 4^{\frac{1}{2}}$
$= \log 2$

Ex. $\int_0^1 [t e^{t^2}\bar{i} + e^{-t}\bar{j} + \bar{k}]\,dt$

$= \int_0^1 t\, e^{t^2}\,\bar{i}\,dt + \int_0^1 e^{-t}\,\bar{j} + \int_0^1 \bar{k}\,dt$

put $t^2 = \theta$.

$2t\,dt = d\theta \Rightarrow t\,dt = \frac{1}{2}d\theta$

if $t=0 \Rightarrow \theta = 0$ & if $t=1 \Rightarrow \theta = 1$

$= \int_0^1 e^{\theta}\,\frac{1}{2}d\theta\,\bar{i} + [-e^{-t}\,\bar{j}]_0^1 + [t\,\bar{k}]_0^1$

$= \frac{1}{2}[e^{\theta}]_0^1\,\bar{i} + [-e^{-1} + e^{0}]\,\bar{j} + [1-0]\,\bar{k}$

$= \frac{1}{2}[e^{1} - e^{0}]\,\bar{i} + [1 - e^{-1}]\,\bar{j} + \bar{k}$

$= \frac{1}{2}(e-1)\bar{i} + (1-e^{-1})\bar{j} + \bar{k}$.

Ex. solve the initial value problem for $\bar{r}(t)$ where $\frac{d}{dt}\bar{r}(t) = 180t\,\bar{i} + (180t - 16t^2)\bar{j}$ with initial condition $\bar{r}(0) = 100\bar{j}$.

$\Rightarrow \int \frac{d}{dt}\bar{r}(t) = \int [180t\,\bar{i} + (180t - 16t^2)\bar{j}]\,dt$

$\bar{r}(t) = 180\frac{t^2}{2}\bar{i} + (180\frac{t^2}{2} - 16\frac{t^3}{3})\bar{j} + c$ —(1)

since $\bar{r}(0) = 100\bar{j}$

$\therefore$ put $t=0$ in (1)

$\bar{r}(0) = 0 + 0 + c$

$\therefore 100 = c \Rightarrow c = 100$

from eqn (1)

$\bar{r}(t) = 90t^2\,\bar{i} + (90t^2 - \frac{16}{3}t^3)\bar{j} + 100$

Ex. Solve the initial value problem for $\bar{r}(t)$ where $\frac{d^2}{dt^2}\bar{r}(t) = -\bar{i} - \bar{j} - \bar{k}$, with initial condition $\bar{r}(0) = 10\bar{i} + 10\bar{j} + 10\bar{k}$ & $\frac{d}{dt}\bar{r}(0) = \bar{0}$

$\Rightarrow \frac{d^2}{dr^2}\bar{r}(t) = -\bar{i} - \bar{j} - \bar{k}$

$\frac{d\bar{r}}{dt} = \int \frac{d^2}{dr^2}\bar{r}(t) = -t\bar{i} - t\bar{j} - t\bar{k} + c$

$\frac{d}{dt}\bar{r}(0) = c$

sind $\frac{d}{dt}\bar{r}(0) = 0$ (given)

$\therefore c = 0$

$\therefore \frac{d\bar{r}}{dt} = -t\bar{i} - t\bar{j} - t\bar{k}$

$\bar{r}(t) = \int(-t\bar{i} - t\bar{j} - t\bar{k})\,dt$

$\bar{r}(t) = -\frac{t^2}{2}\bar{i} - \frac{t^2}{2}\bar{j} - \frac{t^2}{2}\bar{k} + c$ —(1)

$$\bar{r}(0) = c$$

since $\bar{r}(0) = 10\bar{i} + 10\bar{j} + 10\bar{k}$ (given)

$$\therefore \quad c = 10\bar{i} + 10\bar{j} + 10\bar{k}$$

from ①

$$\bar{r}(t) = -\frac{t^2}{2}\bar{i} - \frac{t^2}{2}\bar{j} - \frac{t^2}{2}\bar{k} + 10\bar{i} + 10\bar{j} + 10\bar{k}$$

$$= \left(10 - \frac{t^2}{2}\right)\bar{i} + \left(10 - \frac{t^2}{2}\right)\bar{j} + \left(10 - \frac{t^2}{2}\right)\bar{k}$$

Ex. Solve the initial value problem of $\bar{r}(t)$ where $\frac{d}{dt}\bar{r}(t) = (t^3 + 4t)\bar{i} + t\bar{j} + 2t^2\bar{k}$ with initial condition $\bar{r}(0) = \bar{i} + \bar{j}$

$$\Rightarrow \quad \frac{d}{dt}\bar{r}(t) = (t^3 + 4t)\bar{i} + t\bar{j} + 2t^2\bar{k}$$

$$\bar{r}(t) = \int [(t^3 + 4t)\bar{i} + t\bar{j} + 2t^2 k]\, dt$$

$$\bar{r}(t) = \left[\frac{t^4}{4} + 4\frac{t^2}{2}\right]\bar{i} + \frac{t^2}{2}\bar{j} + 2\frac{t^3}{3}\bar{k} + c \quad \text{—①}$$

$$\bar{r}(0) = c$$

$$\& \quad \bar{r}(0) = \bar{i} + \bar{j} \quad \text{(given)}$$

$$\Rightarrow \quad c = \bar{i} + \bar{j}$$

from ①

$$\bar{r}(t) = \left[\frac{t^4}{4} + 2t^2 + 1\right]\bar{i} + \left[\frac{t^2}{2} + 1\right]\bar{j} + \frac{2t^3}{3}\bar{k}$$

*** Arc Length -**

Arc length refers to the distance between two points along a curves section.

*** Arc length along a space curve -**

The length of smooth curve $\bar{r}(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$, $a \le t \le b$ that is traced exactly once at t increase from $t=a$ to $t=b$ is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

We know that $\bar{v}(t) = \frac{d\bar{r}}{dt} = \frac{dx}{dt}\bar{i} + \frac{dy}{dt}\bar{j} + \frac{dz}{dt}\bar{k}$

$$\therefore \quad |\bar{v}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$\therefore$ Arc length along a space curve is also written as

$$\boxed{L = \int_a^b |\bar{v}(t)|\, dt}$$

* Arc Length Parameter –

The arc length parameter of a smooth curve $\bar{r}(t) = x(t)\,\bar{i} + y(t)\,\bar{j} + z(t)\,\bar{k}$, $t_0 \le t \le t_1$ from the point where $t = t_0$ is

$$s(t) = \int_{t_0}^{t_1} |\bar{v}(\tau)|\, d\tau$$

* Speed on a smooth curve –

If a curve $\bar{r}(t)$ is given in terms of some parameter t and $s(t)$ is the arc length function then the speed with which a particle moves along its path is the magnitude of $\bar{v}$

i.e.

$$\boxed{\frac{ds}{dt} = |\bar{v}(t)|}$$

* Unit tangent vector –

Let $\bar{r}(t) = x(t)\,\bar{i} + y(t)\,\bar{j} + z(t)\,\bar{k}$ be a vector function of a curve in space, then the velocity vector $\bar{v}(t) = \frac{d}{dt}\bar{r}(t)$ is tangent to the curve and the vector

$$\boxed{\bar{T}(t) = \frac{\bar{v}(t)}{|\bar{v}(t)|}}$$

is a unit vector tangent to the curve.

We have

$$\frac{d\bar{r}}{ds} = \frac{d\bar{r}}{dt}\frac{dt}{ds} = \bar{v}(t) \cdot \frac{1}{|\bar{v}(t)|} = \bar{T}$$

This eqn says that $\frac{d\bar{r}}{ds}$ is the unit tangent vector in the direction of the velocity $\bar{v}$.

The unit tangent vector of a smooth curve $\bar{r}(t)$ is

$$\boxed{\bar{T} = \frac{d\bar{r}}{ds} = \frac{\frac{d\bar{r}}{dt}}{\frac{ds}{dt}}}$$

Ex. Find the arc length along the curve $\bar{r}(t) = 6\sin 2t\,\bar{i} + 6\cos 2t\,\bar{j} + 5t\,\bar{k}$ from $t=0$ to $t=\pi$. Also find the unit tangent vector of the curve

$\Rightarrow$ $\bar{r}(t) = 6\sin 2t\,\bar{i} + 6\cos 2t\,\bar{j} + 5t\,\bar{k}$

$$\bar{v}(t) = \frac{d\bar{r}}{dt} = 12\cos 2t\,\bar{i} - 12\sin 2t\,\bar{j} + 5\bar{k}$$

If $\bar{n} = a\bar{i} + b\bar{j} + c\bar{k}$ then $|\bar{n}| = \sqrt{a^2+b^2+c^2}$

$$\therefore |\bar{v}(t)| = \sqrt{144\cos^2 2t + 144\sin^2 2t + 25}$$

$$= \sqrt{144(\cos^2 2t + \sin^2 2t) + 25}$$

$$= \sqrt{144+25} = \sqrt{169} = 13$$

Arc length

$$L = \int_0^{\pi} |\bar{v}(t)|\,dt$$

$$= \int_0^{\pi} 13\,dt$$

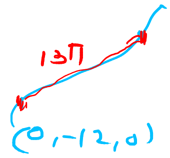
$$= [13t]_{t=0}^{\pi} = 13\pi$$

Unit tgt vector

$$\bar{T} = \frac{\bar{v}(t)}{|\bar{v}(t)|}$$

$$= \frac{1}{13}(12\cos 2t\,\bar{i} - 12\sin 2t\,\bar{j} + 5\bar{k})$$

Ex. Find the point on the curve $\bar{r}(t) = 12\sin t\,\bar{i} - 12\cos t\,\bar{j} + 5t\,\bar{k}$ at a distance 13π units along the curve from the point $(0,-12,0)$ in the direction of increasing arc length.



$$\Rightarrow \quad \bar{r}(t) = 12\sin t\,\bar{i} - 12\cos t\,\bar{j} + 5t\,\bar{k}$$

$$\bar{v}(t) = \frac{d\bar{r}}{dt} = 12\cos t\,\bar{i} + 12\sin t\,\bar{j} + 5\bar{k}$$

$$|\bar{v}(t)| = \sqrt{144\cos^2 t + 144\sin^2 t + 25} = 13$$

We know that arc length is given by

$$L = \int_{t_0}^{t_1} |\bar{v}(t)|\,dt$$

$\bar{r}(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$

starting point is $(0,-12,0)$

i.e. $x(t) = 0$, $y(t) = -12$, $z(t) = 0$

$$\Rightarrow 12\sin t = 0, \quad -12\cos t = -12, \quad 5t = 0$$

$$\Rightarrow t = 0 \quad \text{(lower limit)}$$

$$\therefore L = \int_{0}^{t_1} |\bar{v}(t)|\,dt$$

$$13\pi = \int_0^{t_1} 13\,dt = 13[t]_0^{t_1} = 13t_1$$

$$\Rightarrow 13\pi = 13t_1$$

$$\Rightarrow t_1 = \pi$$

at $t = \pi$

$$\therefore x(t) = 12\sin t \Rightarrow x(\pi) = 12\sin\pi = 0$$

$$y(t) = -12\cos t \Rightarrow y(\pi) = -12\cos\pi = 12$$

$$z(t) = 5t \Rightarrow z(\pi) = 5\pi$$

$\therefore$ The required point is $(0, 12, 5\pi)$

Ex. Find the point on the curve $\bar{r}(t) = 5\sin t\,\bar{i} + 5\cos t\,\bar{j} + 12t\,\bar{k}$ at a distance 26π units along the curve from the point $(0,5,0)$ in the direction of increasing arc length.

$\Rightarrow \quad \bar{r}(t) = 5\sin t\,\bar{i} + 5\cos t\,\bar{j} + 12t\,\bar{k} \quad \text{—}①$

$\bar{v}(t) = 5\cos t\,\bar{i} - 5\sin t\,\bar{j} + 12\,\bar{k}$

$|\bar{v}(t)| = \sqrt{25\cos^2 t + 25\sin^2 t + 144}$

$= \sqrt{25 + 144} = \sqrt{169} = 13$

$$L = \int_{t_0}^{t_1} |\bar{v}(t)|\,dt$$

starting point is $(0, 5, 0)$

$x = 0,\ y = 5,\ z = 0$

also from eqn ①

$x = 5\sin t,\quad y = 5\cos t,\ z = 12t$

$\Rightarrow 5\sin t = 0,\quad 5\cos t = 5,\ 12t = 0$

$\Rightarrow t = 0$

$$\therefore\ L = \int_0^{t_1} |\bar{v}(t)|\,dt$$

$$26\pi = \int_0^{t_1} 13\,dt = \Big[13t\Big]_{t=0}^{t_1} = 13t_1$$

$\Rightarrow t_1 = 2\pi$

$\sin n\pi = 0$

$\cos n\pi = (-1)^n$

at $t = 2\pi$

$x(t) = 5\sin t \Rightarrow 5\sin 2\pi = 0$

$y(t) = 5\cos t \Rightarrow 5\cos 2\pi = 1$

$z(t) = 12t \Rightarrow 12(2\pi) = 24$

$\therefore$ Required point is

$(0, 1, 24)$

Ex. Find the arc length parameter along the curve $\bar{r}(t) = e^t\cos t\,\bar{i} + e^t\sin t\,\bar{j} + e^t\,\bar{k}$ from the point where $t = 0$.

$\Rightarrow \quad \bar{v}(t) = \dfrac{d\bar{r}}{dt} = [-e^t\sin t + \cos t \cdot e^t]\,\bar{i} + [e^t\cos t + \sin t \cdot e^t]\,\bar{j} + e^t\,\bar{k}$

$\bar{v}(t) = e^t(\cos t - \sin t)\,\bar{i} + e^t(\cos t + \sin t)\,\bar{j} + e^t\,\bar{k}$

$$\therefore |\bar{v}(t)| = \sqrt{e^{2t}(\cos^2 t - 2\cos t \sin t + \sin^2 t) + e^{2t}(\cos^2 t + 2\cos t \sin t + \sin^2 t) + e^{2t}}$$

$$= \sqrt{e^{2t}(1) + e^{2t}(1) + e^{2t}}$$

$$= \sqrt{3e^{2t}} = \sqrt{3}\, e^t$$

Arc length parameter

$$s(t) = \int_0^t |\bar{v}(t)|\,dt$$

$$= \int_0^t \sqrt{3}\, e^t\, dt$$

$$= \sqrt{3}\,[e^t]_{t=0}^{t}$$

$$= \sqrt{3}\,[e^t - e^0] = \sqrt{3}\,(e^t - 1)$$

Ex. Find the arc length parameter along the curve $\bar{r}(t) = (\cos t + t\sin t)\,\bar{i} + (\sin t - t\cos t)\,\bar{j}$ from the point where $t = 0$.

$\Rightarrow$
$$\bar{v}(t) = \frac{d\bar{r}}{dt} = [-\sin t + t\cos t + \sin t]\,\bar{i} + [\cos t + t\sin t - \cos t]\,\bar{j}$$

$$= t\cos t\,\bar{i} + t\sin t\,\bar{j}$$

$$|\bar{v}(t)| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = \sqrt{t^2} = t$$

Arc length parameter

$$s = \int_0^t |\bar{v}(t)|\,dt$$

$$= \int_0^t t\,dt = \left[\frac{t^2}{2}\right]_0^t = \frac{t^2}{2}$$

Ex. Find the arc length along the curve $\bar{r}(t) = \sqrt{2}t\,\bar{i} + \sqrt{2}t\,\bar{j} + (1-t^2)\,\bar{k}$ from $(0,0,1)$ to $(\sqrt{2}, \sqrt{2}, 0)$.

$\Rightarrow$ $x(t) = \sqrt{2}t$, $y(t) = \sqrt{2}t$, $z(t) = 1 - t^2$

at $(0,0,1)$

$0 = \sqrt{2}t$, $0 = \sqrt{2}t$, $1 = 1 - t^2$

$\Rightarrow t = 0$

at $(\sqrt{2}, \sqrt{2}, 0)$

$x = \sqrt{2}t \Rightarrow \sqrt{2} = \sqrt{2}t \Rightarrow t = 1$

$\bar{v}(t) = \frac{d\bar{r}}{dt} = \sqrt{2}\,\bar{i} + \sqrt{2}\,\bar{j} - 2t\,\bar{k}$

$|\bar{v}(t)| = \sqrt{2+2+4t^2} = \sqrt{4(1+t^2)} = 2\sqrt{1+t^2}$

arc length is

$L = \int_0^1 |\bar{v}(t)|\,dt$

$\int \sqrt{a^2+x^2} = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log(x+\sqrt{a^2+x^2})$

$= \int_0^1 2\sqrt{1+t^2}\,dt$

$a=1,\ x=t$

$= \cancel{2}\left[\frac{t}{\cancel{2}}\sqrt{1+t^2} + \frac{1}{\cancel{2}}\log(t+\sqrt{1+t^2})\right]_{t=0}^{1}$

$= [1\sqrt{2} + \log(1+\sqrt{2})]$

$\quad - [0 + \log(0+\sqrt{1})]$

$= \sqrt{2} + \log(1+\sqrt{2}) - \underbrace{\log(1)}_{0}$

$L = \sqrt{2} + \log(1+\sqrt{2})$.

* Curvature and Normal vector -

1) Curvature of a plane curve -

If $\bar{r}(t)$ is smooth curve. If $\bar{T}$ is the unit vector of a smooth curve then curvature of a curve is

$$\boxed{K = \frac{1}{|\bar{v}(t)|}\left|\frac{d\bar{T}}{dt}\right|}$$

where $\bar{T} = \frac{\bar{v}(t)}{|\bar{v}(t)|}$

2) Principal unit normal vector is $\bar{N}$

$$\boxed{\bar{N} = \frac{\frac{d\bar{T}}{dt}}{\left|\frac{d\bar{T}}{dt}\right|}}$$

3] Binormal vector is

$$\boxed{\bar{B} = \bar{T} \times \bar{N}}$$

Ex. Find $\bar{T}$, $\bar{N}$ and k for the plane curve

$\bar{r}(t) = \log(\sec t)\,\bar{i} + t\,\bar{j}$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$

$\Rightarrow \quad \bar{r}(t) = \log(\sec t)\,\bar{i} + t\,\bar{j}$

$\Rightarrow \quad \bar{v}(t) = \frac{d\bar{r}}{dt} = \frac{1}{\sec t} \cdot \sec t \cdot \tan t\,\bar{i} + \bar{j}$

$\bar{v}(t) = \tan t\,\bar{i} + \bar{j}$

$|\bar{v}(t)| = \sqrt{\tan^2 t + 1} = \sqrt{\sec^2 t} = \sec t$

$$\bar{T} = \frac{\bar{v}}{|\bar{v}|}$$

$\bar{T} = \frac{1}{\sec t}[\tan t\,\bar{i} + \bar{j}] = \cos t[\tan t\,\bar{i} + \bar{j}]$

$= \frac{\sin t}{\cos t} \cdot \cos t\,\bar{i} + \cos t\,\bar{j}$

$\bar{T} = \sin t\,\bar{i} + \cos t\,\bar{j}$

$\frac{d\bar{T}}{dt} = \cos t\,\bar{i} - \sin t\,\bar{j}$

$\Rightarrow \quad \left|\frac{d\bar{T}}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t} = 1$

$$k = \frac{1}{|\bar{v}|}\left|\frac{d\bar{T}}{dt}\right| = \frac{1}{\sec t}(1) = \cos t$$

$$\bar{N} = \frac{\frac{d\bar{T}}{dt}}{\left|\frac{d\bar{T}}{dt}\right|} = \frac{1}{1}[\cos t\,\bar{i} - \sin t\,\bar{j}]$$

$\bar{N} = \cos t\,\bar{i} - \sin t\,\bar{j}$

Ex. Find $\bar{T}$, $\bar{N}$ & k for the plane curve

$\bar{r}(t) = (2t+3)\,\bar{i} + (5-t^2)\,\bar{j}$

$$\Rightarrow \quad \bar{v}(t) = \frac{d\bar{r}}{dt} = 2\bar{i} - 2t\,\bar{j}$$

$$|\bar{v}(t)| = \sqrt{4+4t^2} = 2\sqrt{1+t^2}$$

$$\bar{T} = \frac{\bar{v}}{|\bar{v}|}$$

$$\bar{T} = \frac{1}{2\sqrt{1+t^2}}\left[2\bar{i} - 2t\,\bar{j}\right]$$

$$\bar{T} = \frac{1}{\sqrt{1+t^2}}\,\bar{i} - \frac{t}{\sqrt{1+t^2}}\,\bar{j}$$

$$\frac{u}{v} = \frac{vu' - uv'}{v^2}$$

$$\frac{d\bar{T}}{dt} = \left[\frac{\sqrt{1+t^2}\,(0) - 1\,\frac{1}{\cancel{2}\sqrt{1+t^2}}\,\cancel{2}t}{1+t^2}\right]\bar{i}$$

$$(1+t^2)'\ (1+t)^{\frac{1}{2}}$$

$$- \left[\frac{\sqrt{1+t^2}\,(1) - t\,\frac{1}{\cancel{2}\sqrt{1+t^2}}\,\cancel{2}t}{1+t^2}\right]\bar{j}$$

$$= -\frac{t}{(1+t^2)^{3/2}}\,\bar{i} - \left[\frac{1+t^2-t^2}{(1+t^2)^{3/2}}\right]\bar{j}$$

$$\frac{d\bar{T}}{dt} = \frac{-t}{(1+t^2)^{3/2}}\,\bar{i} - \frac{1}{(1+t^2)^{3/2}}\,\bar{j}$$

$$\left|\frac{d\bar{T}}{dt}\right| = \sqrt{\frac{t^2}{(1+t^2)^3} + \frac{1}{(1+t^2)^3}}$$

$$= \sqrt{\frac{1+t^2}{(1+t^2)^3}} = \sqrt{\frac{1}{(1+t^2)^2}}$$

$$= \frac{1}{1+t^2}$$

$$\left|\frac{d\bar{T}}{dt}\right| = \frac{1}{1+t^2}$$

$$\kappa = \frac{1}{|\bar{v}|}\left|\frac{d\bar{T}}{dt}\right| = \frac{1}{2\sqrt{1+t^2}} \cdot \frac{1}{1+t^2} = \frac{1}{2(1+t^2)^{3/2}}$$

$$\bar{N} = \frac{\frac{d\bar{T}}{dt}}{\left|\frac{d\bar{T}}{dt}\right|}$$

$$\bar{N} = \frac{1}{\frac{1}{1+t^2}} \cdot \left[\frac{-t}{(1+t^2)^{3/2}} \bar{i} - \frac{1}{(1+t^2)^{3/2}} \bar{j} \right]$$

$$= (1+t^2) \left[-\frac{t}{(1+t^2)^{3/2}} \bar{i} - \frac{1}{(1+t^2)^{3/2}} \bar{j} \right]$$

$$\bar{N} = \frac{-t}{\sqrt{1+t^2}} \bar{i} - \frac{1}{\sqrt{1+t^2}} \bar{j}$$

Ex. Find the curvature of $\bar{r}(t) = (\cos 2t, -\sin 2t, 4t)$

$\Rightarrow$ $\bar{r}(t) = \cos 2t \, \bar{i} - \sin 2t \, \bar{j} + 4t \, \bar{k}$

$$\bar{v}(t) = \frac{d\bar{r}}{dt} = -2\sin 2t \, \bar{i} - 2\cos 2t \, \bar{j} + 4\bar{k}$$

$$|\bar{v}(t)| = \sqrt{4\sin^2 2t + 4\cos^2 2t + 16}$$

$$= \sqrt{4+16} = \sqrt{20} = \sqrt{4 \times 5} = 2\sqrt{5}$$

$$\bar{T} = \frac{\bar{v}}{|\bar{v}|}$$

$$= \frac{1}{2\sqrt{5}} (-2\sin 2t \, \bar{i} - 2\cos 2t \, \bar{j} + 4\bar{k})$$

$$\frac{d\bar{T}}{dt} = \frac{1}{2\sqrt{5}} [-4\cos 2t \, \bar{i} + 4\sin 2t \, \bar{j} + 0\bar{k}]$$

$$\left|\frac{d\bar{T}}{dt}\right| = \frac{1}{2\sqrt{5}} \sqrt{16\cos^2 2t + 16\sin^2 2t}$$

$$= \frac{1}{2\sqrt{5}} \sqrt{16} = \frac{1}{2\sqrt{5}} \cdot 4 = \frac{2}{\sqrt{5}}$$

Ex. Find the unit tangent vector for the vector function $\bar{r}(t) = (t^2+1, 3-t, t^3)$

$\Rightarrow$ $\bar{r}(t) = (t^2+1)\bar{i} + (3-t)\bar{j} + t^3 \bar{k}$

$$\bar{v}(t) = 2t\bar{i} - \bar{j} + 3t^2 \bar{k}$$

$$|\bar{v}(t)| = \sqrt{4t^2+1+9t^4}$$

unit tgt vector

$$\bar{T} = \frac{\bar{v}}{|\bar{v}|}$$

$$= \frac{1}{\sqrt{4t^2+1+9t^4}}\left[2t\bar{i} - \bar{j} + 3t^2\bar{k}\right]$$

Ex. Find the unit normal and the binormal vectors for the following vector function.

$$\bar{r}(t) = (\cos 2t, \sin 2t, 3)$$

$$\Rightarrow \quad \bar{r}(t) = \cos 2t\,\bar{i} + \sin 2t\,\bar{j} + 3\bar{k}$$

$$\bar{v}(t) = -2\sin 2t\,\bar{i} + 2\cos 2t\,\bar{j} + 0\bar{k}$$

$$|\bar{v}(t)| = \sqrt{4\sin^2 2t + 4\cos^2 2t + 0}$$

$$= \sqrt{4(1)+0} = 2$$

$$\bar{T} = \frac{\bar{v}(t)}{|\bar{v}(t)|}$$

$$= \frac{1}{2}\left[-2\sin 2t\,\bar{i} + 2\cos 2t\,\bar{j} + 0\bar{k}\right]$$

$$\bar{T} = -\sin 2t\,\bar{i} + \cos 2t\,\bar{j} + 0\bar{k}$$

$$\frac{d\bar{T}}{dt} = -2\cos 2t\,\bar{i} - 2\sin 2t\,\bar{j} + 0\bar{k}$$

$$\left|\frac{d\bar{T}}{dt}\right| = 2$$

Normal vector

$$\bar{N} = \frac{d\bar{T}/dt}{\left|\frac{d\bar{T}}{dt}\right|}$$

$$= \frac{1}{2}\left[-2\cos 2t\,\bar{i} - 2\sin 2t\,\bar{j} + 0\bar{k}\right]$$

$$\bar{N} = -\cos 2t\,\bar{i} - \sin 2t\,\bar{j} + 0\bar{k}$$

Binormal vector –

$$\bar{B} = \bar{T} \times \bar{N}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -\sin 2t & \cos 2t & 0 \\ -\cos 2t & -\sin 2t & 0 \end{vmatrix}$$

$$= \bar{i}(0) - \bar{j}(0) + \bar{k}(\sin^2 2t + \cos^2 2t)$$

$$\bar{B} = \bar{k}$$

Reference: Vector Calculus, Text for S.Y.B.Sc., Golden series by Nirali Prakashan.