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Chapter 2: Interpolation

Topic- Finite Difference Operators and their relations, Newton's Interpolation Formulae Lagrange's Interpolation Formula

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Chapter - 2
Interpolation

* Finite difference operators & their relations

1] Forward Differences or Leading diff. —

Let $y = f(x)$ be a single valued f^n & conti on the interval $[x_0, x_n]$. Let values of x are $x_0, x_0+h, x_0+2h, \dots, x_0+nh = x_n$ & corresponding values of y are $y_0 = f(x_0)$, $y_1 = f(x_0+h)$, $y_2 = f(x_0+2h)$ $\dots, y_n = f(x_0+nh)$

then 1st forward difference is defined as

$$\Delta f(x) = f(x+h) - f(x)$$

The symbol Δ is used for forward difference operator

$$\Delta f(x_0) = f(x_0+h) - f(x_0)$$

$$\therefore \Delta y_0 = y_1 - y_0$$

$$\Delta y_5 = y_6 - y_5$$

$$\Delta y_8 = y_9 - y_8$$

III^{ly}

$$\Delta y_1 = y_2 - y_1$$

$$\Delta y_2 = y_3 - y_2$$

$$\vdots$$

$$\Delta y_{n-1} = y_n - y_{n-1}$$

The second forward difference operator is defined by

$$\Delta^2 f(x) = \Delta(\Delta f(x))$$

$$= \Delta(f(x+h) - f(x))$$

$$= \Delta f(x+h) - \Delta f(x)$$

$$= f(x+2h) - f(x+h) - f(x+h) + f(x)$$

$$\Delta^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$$

$$\therefore \Delta^2 f(x_0) = f(x_0+2h) - 2f(x_0+h) + f(x_0)$$

$$\Delta^2 y_0 = y_2 - 2y_1 + y_0$$

or

$$\Delta^2 y_0 = \Delta(\Delta y_0)$$

$$= \Delta(y_1 - y_0)$$

$$= \Delta y_1 - \Delta y_0$$

$$\Delta^2 y_2 = \Delta(\Delta y_2)$$

$$= \Delta(y_3 - y_2)$$

$$= \Delta y_3 - \Delta y_2$$

$$= y_4 - y_3 - y_3 + y_2$$

$$= y_2 - y_1 - y_1 + y_0$$

$$\Delta^2 y_0 = y_2 - 2y_1 + y_0$$

$$= y_4 - 2y_3 + y_2$$

Remark – Δ satisfies following properties

1] The difference of const function is zero.

i.e. if $f(x) = c$ then $\Delta f(x) = 0$

$\left[\because f(x) = c \right.$
$\Delta f(x) = f(x+h) - f(x)$
$= c - c$
$= 0$

2] $\Delta [f(x) \pm g(x)] = \Delta f(x) \pm \Delta g(x)$

3] $\Delta^m [\Delta^n f(x)] = \Delta^{m+n} f(x)$

$a^m \cdot a^n = a^{m+n}$

* Forward diff. table

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0			
		$y_1 - y_0 = \Delta y_0$		
x_1	y_1		$\Delta y_1 - \Delta y_0 = \Delta^2 y_0$	
		$y_2 - y_1 = \Delta y_1$		$\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0$
x_2	y_2		$\Delta y_2 - \Delta y_1 = \Delta^2 y_1$	
		$y_3 - y_2 = \Delta y_2$		
x_4	y_3			

Ex. construct the forward difference table for the following set of values

$f(x) = x^3 + 5x - 7$, $x = -1, 0, 1, 2, 3, 4, 5$

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-1	-13				
		$-7-(-13) = 6$			
0	-7		0		
		$-1-(-7) = 6$		6	
1	-1		6		0
		$11-(-1) = 12$		6	
2	11		12		0
		24		6	
3	35		18		0
		42		6	
4	77		24		
		66			
5	143				

2) For the given set of values, form the forward diff. table and write the values of $\Delta^2 y_{10}$, $\Delta^3 y_{15}$, $\Delta^5 y_{10}$

x	10	15	20	25	30	35
y	19.97	21.51	22.47	23.52	24.65	25.89

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
10	19.97					
		$1.54 = \Delta y_{10}$				
15	21.51		$\Delta^2 y_{10}$ -0.58			
		0.96 Δy_{15}		$\Delta^3 y_{10}$ 0.67		
20	22.47		$\Delta^2 y_{15}$ 0.09		$\Delta^4 y_{10}$ -0.68	
		1.05 Δy_{20}		$\Delta^3 y_{15}$ -0.01		$0.72 = \Delta^5 y_{10}$
25	23.52		$\Delta^2 y_{20}$ 0.08		$\Delta^4 y_{15}$ 0.04	
		1.13 Δy_{25}		$\Delta^3 y_{20}$ 0.03		
30	24.65		$\Delta^2 y_{25}$ 0.11			
		1.24 Δy_{35}				
35	25.89					

$$\Delta^2 y_{10} = -0.58, \quad \Delta y_{20} = 1.05, \quad \Delta^3 y_{15} = -0.01,$$

$$\Delta^5 y_{10} = 0.72$$

Ex. Construct the forward diff table for the following set of values of $f(x) = x^2+x+1$, $x = -2, -1, 0, 1, 2$

Find the value of $f(3)$

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
-2	3			
		-2		
-1	1		2	
		0		0
0	1		2	
		2		0
1	3		2	
		4		$a-13$
2	7		$a-11$	
		$a-7$		
3	a			

Take $a-13 = 0 \Rightarrow a = 13$

Backword Differences –

Let $y = f(x)$ be a single valued function then backword diff is given by

$$\nabla f(x) = f(x) - f(x-h)$$

$$\nabla^2 f(x) = \nabla(\nabla f(x))$$
$$= \nabla(f(x) - f(x-h))$$
$$= \nabla f(x) - \nabla f(x-h)$$
$$= f(x) - f(x-h) - f(x-h) + f(x-2h)$$
$$= f(x) - 2f(x-h) + f(x-2h)$$

$\Delta f(x) = f(x+h) - f(x)$ forward diff

or

$$\nabla y_1 = y_1 - y_0$$
$$\nabla y_2 = y_2 - y_1$$
$$\vdots$$
$$\nabla y_n = y_n - y_{n-1}$$

$\Delta y_1 = y_2 - y_1$

$\Delta y_2 = y_3 - y_2$

$$\nabla^2 y_2 = \nabla(\nabla y_2)$$
$$= \nabla(y_2 - y_1)$$
$$= \nabla y_2 - \nabla y_1$$
$$= (y_2 - y_1) - (y_1 - y_0)$$
$$= y_2 - 2y_1 + y_0$$

Ex. construct a backward diff table for $y = \log x$

x	10	20	30	40	50
y	1	1.3010	1.4771	1.6021	1.6990

Find the values of $\nabla^2 \log 40$, $\nabla^3 \log 50$

Backward diff table

x	y	∇	∇^2	∇^3	
x_0	y_0				
		$y_1 - y_0$ ∇y_1			
x_1	y_1		$\nabla^2 y_2$		
		$y_2 - y_1 = \nabla y_2$		$\nabla^3 y_3$	
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$
		$y_3 - y_2 = \nabla y_3$		$\nabla^3 y_4$	
x_3	y_3		$\nabla^2 y_4$		
x_4	y_4	∇y_4			

Sol^n.

x	$f(x) = \log x$	∇	∇^2	∇^3	∇^4
10	1				
		0.3010 ($\nabla \log 20$)			
20	1.3010		−0.1249 ($\nabla^2 \log 30$)		
		0.1761 ($\nabla \log 30$)		0.0738 ($\nabla^3 \log 40$)	
30	1.4771		−0.0511 ($\nabla^2 \log 40$)		−0.0508 ($\nabla^4 \log 50$)
		0.1250 ($\nabla \log 40$)		0.023 ($\nabla^3 \log 50$)	
40	1.6021		−0.0281 ($\nabla^2 \log 50$)		
		0.0969 ($\nabla \log 50$)			
50	1.6990				

$\nabla^2 \log 40 = -0.0511$, $\nabla^3 \log 50 = 0.023$

Ex. Prepare a backword diff table for the function $f(x) = x^3 + 5x - 7$, for $x = 0, 1, 2, 3, 4, 5$, and find $f(-1)$

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
−1	a				
		−7−a			
0	−7		13+a		
		6		−7−a	
1	−1		6		13+a
		12		6	
2	11		12		0
		24		6	
3	35		18		0
		42		6	
4	77		24		
		66			
5	143				

$6 - (-7-a)$

$13 + a$

$6 - (13+a)$

$-7 - a$

$6 - (-7-a)$

$13 + a$

$\therefore 13 + a = 0$

$\Rightarrow a = -13$

$\therefore f(-1) = -13$

* The shift operator

The shift operator is denoted by E & is given by

$$Ef(x) = f(x+h)$$

$$E^2 f(x) = E(E f(x))$$

$$= E(f(x+h))$$

$$E^2 f(x) = f(x+2h)$$

$$E^3 f(x) = f(x+3h)$$

$$\vdots$$

$$E^n f(x) = f(x+nh)$$

$$E^{-1} f(x) = f(x-h)$$

* Relation betn Δ, ∇, & E.

$\Delta f(x) = f(x+h) - f(x)$

$\nabla f(x) = f(x) - f(x-h)$

$E f(x) = f(x+h)$

1] S.T. $E \equiv 1+\Delta$ or $\Delta \equiv E-1$

consider
$$E f(x) = f(x+h)$$

consider
$$(1+\Delta) f(x) = f(x) + \Delta f(x)$$

$$= f(x) + f(x+h) - f(x)$$

$$(1+\Delta) f(x) = f(x+h)$$

$$\therefore \quad E f(x) = (1+\Delta) f(x)$$

$$E \equiv 1+\Delta$$

or $\quad E-1 \equiv \Delta$

2] $\nabla \equiv 1 - E^{-1}$ or $E \equiv (1-\nabla)^{-1}$

$$\Rightarrow \quad \nabla f(x) = f(x) - f(x-h)$$

consider $(1-E^{-1}) f(x) = f(x) - E^{-1} f(x)$

$$= f(x) - f(x-h)$$

$$\nabla f(x) = (1 - E^{-1}) f(x)$$

$$\nabla \equiv (1 - E^{-1})$$

$$E^{-1} \equiv 1 - \nabla$$

$$\frac{1}{E} \equiv 1 - \nabla$$

$$E \equiv \frac{1}{1-\nabla} = (1-\nabla)^{-1}$$

3] $E\nabla \equiv \nabla E \equiv \Delta$

consider

$$(E\nabla) f(x) = E(\nabla f(x))$$
$$= E(f(x) - f(x-h))$$
$$= E f(x) - E f(x-h)$$
$$= f(x+h) - f(x-h+h)$$
$$(E\nabla) f(x) = f(x+h) - f(x) \quad \text{--- } ①$$

consider $(\nabla E) f(x) = \nabla(E f(x))$
$$= \nabla f(x+h)$$
$$= f(x+h) - f(x) \quad \text{--- } ②$$

$$\Delta f(x) = f(x+h) - f(x) \quad \text{--- } ③$$

from ①, ② & ③

$$\therefore \ E\nabla \equiv \Delta E \equiv \Delta$$

4] $\Delta \cdot \nabla \equiv \Delta - \nabla$

$$\Rightarrow \quad (\Delta \cdot \nabla) f(x) = \Delta(\nabla f(x))$$
$$= \Delta(f(x) - f(x-h))$$
$$= \Delta f(x) - \Delta f(x-h)$$
$$= f(x+h) - f(x) - f(x) + f(x-h)$$
$$= f(x+h) - 2f(x) + f(x-h)$$

$$(\Delta - \nabla) f(x) = \Delta f(x) - \nabla f(x)$$
$$= f(x+h) - f(x) - f(x) + f(x-h)$$
$$= f(x+h) - 2f(x) + f(x-h)$$

$$\therefore \quad \Delta \cdot \nabla \equiv \Delta - \nabla$$

* Fundamental thm of differences of polynomial -
If $f(x)$ is a n^{th} degree polynomial in x, then the n^{th} difference of $f(x)$ is const $\Delta^n f(x) = const$ and $\Delta^{n+1} f(x)$ and all higher differences are zero when the values of the independent variables are at equal interval.

* Technique to determine the missing term -
If n values of $f(x)$ are given then to determine the missing value of $f(x)$, we assume that $f(x)$ is polynomial of degree $(n-1)$

$$\therefore \quad \Delta^{n-1} f(x) = const$$

$$\therefore \quad \Delta^n f(x) = 0$$

since $\quad \Delta = E - 1 \qquad\qquad E = 1 + \Delta$

$$\therefore \quad (E-1)^n f(x) = 0$$

By expanding this we get require missing term.

Ex. Estimate the missing term in the following data

x	0	1	2	3	4
$y = f(x)$	1	3	9	?	81

$\Rightarrow$ Here 4 value of $f(x)$ are given

$$\therefore \quad \Delta^4 f(x) = 0$$

$$(E-1)^4 f(x) = 0$$

$E = 1 + \Delta$
$\Rightarrow \Delta = E - 1$

$$(4c_0 E^4 (1)^0 - 4c_1 E^3 1 + 4c_2 E^2 1^2 - 4c_3 E^1 1^3 + 4c_4 E^0 1) f(x) = 0$$

$(a-b)^4$
$= 4c_0 a^4 b^0 - 4c_1 a^3 b^1$
$+ 4c_2 a^2 b^2$
$- 4c_3 a^1 b^3$
$+ 4c_4 a^0 b^4$

$$(E^4 - 4E^3 + 6E^2 - 4E + 1) f(x) = 0$$

$$E^4 f(x) - 4E^3 f(x) + 6E^2 f(x) - 4E f(x) + f(x) = 0$$

$$f(x+4h) - 4f(x+3h) + 6f(x+2h) - 4f(x+h) + f(x) = 0$$

$4c_2 = \frac{4 \times 3}{2} = 6$

$4c_3 = \frac{4 \times 3 \times 2}{3 \times 2 \times 1} = 4$

here $h = 1$ [diff betn 2 cons. x]

$E f(x) = f(x+h)$

$E^2 f(x) = f(x+2h)$

$E^n f(x) = f(x+nh)$

$f(x+4) - 4f(x+3) + 6f(x+2) - 4f(x+1) + f(x) = 0$

put $x = 0$

$f(4) - 4f(3) + 6f(2) - 4f(1) + f(0) = 0$

$81 - 4f(3) + 6(9) - 4(3) + (1) = 0$

$81 - 4f(3) + 54 - 12 + 1 = 0$

$-4f(3) + 135 - 11 = 0$

$-4f(3) + 124 = 0$

$-4f(3) = -124$

$f(3) = \frac{-124}{-4} = 31$

2] Find the missing terms in the following

x	0	5	10	15	20	25	30
$y = f(x)$	1	3	?	73	225	?	1153

$\Rightarrow$ Here 5 values of $f(x)$ are given

$\therefore$ Take $\Delta^5 f(x) = 0$

$E = 1 + \Delta$

$\Delta = E - 1$

$(E-1)^5 f(x) = 0$

$({}^5C_0 E^5 - {}^5C_1 E^4 + {}^5C_2 E^3 - {}^5C_3 E^2 + {}^5C_4 E - {}^5C_5 1) f(x) = 0$

$(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1) f(x) = 0$

${}^5C_2 = \frac{5 \times 4}{2} = 10$

$E^5 f(x) - 5E^4 f(x) + 10E^3 f(x) - 10E^2 f(x) + 5E f(x) - f(x) = 0$

$$f(x+5h) - 5f(x+4h) + 10f(x+3h) - 10f(x+2h) + 5f(x+h) - f(x) = 0$$

here $h=5$

$$f(x+25) - 5f(x+20) + 10f(x+15) - 10f(x+10) + 5f(x+5) - f(x) = 0 \quad \text{--- } 1$$

put $x=0$ & $x=5$ in 1

$$f(25) - 5f(20) + 10f(15) - 10f(10) + 5f(5) - f(0) = 0$$

$$f(25) - 5(225) + 10(73) - 10f(10) + 5(3) - 1 = 0$$

$$f(25) - 1125 + 730 - 10f(10) + 15 - 1 = 0$$

$$f(25) - 10f(10) = 381 \quad \text{--- } 2$$

put $x=5$ in 1

$$f(30) - 5f(25) + 10f(20) - 10f(15) + 5f(10) - f(5) = 0$$

$$1153 - 5f(25) + 10(225) - 10(73) + 5f(10) - 3 = 0$$

$$-5f(25) + 5f(10) = -2670$$

$$f(25) - f(10) = 534 \quad \text{--- } 3$$

2 − 3

$$-9f(10) = -153$$

$$f(10) = 17$$

put $f(10)$ in 3

$$\therefore f(25) - 17 = 534$$

$$f(25) = 551$$

* Factorial notation –

n^{th} factorial of x is denoted by $x^{(n)}$ & is given by

$$x^{(n)} = x(x-h)(x-2h)\cdots(x-(n-1)h)$$

For natural no $n! = n(n-1)\ldots 2\cdot 1$

$$x^{(0)} = 1$$
$$x^{(1)} = x$$
$$x^{(2)} = x(x-h)$$
$$x^{(3)} = x(x-h)(x-2h)$$
$$x^{(4)} = x(x-h)(x-2h)(x-3h)$$

if $h=1$

$$x^{(2)} = x(x-1)$$
$$x^{(3)} = x(x-1)(x-2)$$
$$x^{(4)} = x(x-1)(x-2)(x-3)$$

* $$\Delta^n x^{(n)} = nh\, x^{(n-1)}$$

$$\frac{d}{dx} x^n = n x^{n-1}$$

$$\Delta x^{(4)} = 4x^{(3)}$$
$$\Delta^2 x^{(4)} = 12 x^{(2)}$$

$$\Delta^m x^{(n)} = 0 \quad , \quad m > n$$

Ex. Express $f(x) = x^3 - 3x^2 - 3x + 5$ and its differences in factorial notation taking $h=1$

$\Rightarrow$ consider $f(x) = x^3 - 3x^2 - 3x + 5$

$$x^3 - 3x^2 - 3x + 5 = Ax(x-1)(x-2) + Bx(x-1) + Cx + D$$
$$= Ax(x^2 - 3x + 2) + B(x^2 - x) + Cx + D$$
$$x^3 - 3x^2 - 3x + 5 = A(x^3 - 3x^2 + 2x) + B(x^2 - x) + Cx + D$$
$$= Ax^3 + (-3A + B)x^2 + (2A - B + C)x + D$$

Equating the coeff.

$x^3 \Rightarrow \quad A = 1$

$x^2 \Rightarrow \quad -3A + B = -3 \quad \Rightarrow -3(1) + B = -3 \Rightarrow B = 0$

$x \Rightarrow \quad 2A - B + C = -3 \Rightarrow 2(1) - 0 + C = -3 \Rightarrow C = -5$

const $\Rightarrow \quad D = 5$

$$\therefore x^3 - 3x^2 - 3x + 5 = x(x-1)(x-2) - 5x + 5$$
$$\therefore f(x) = x^{(3)} - 5x^{(1)} + 5$$
$$\therefore \Delta f(x) = 3x^{(2)} - 5$$
$$\Delta^2 f(x) = 6x^{(1)}$$
$$\Delta^3 f(x) = 6$$
$$\Delta^4 f(x) = 0$$

Ex. $f(x) = 2x^3 - 3x^2 - 3x - 10$ express $f(x)$ & its diff. in factorial notation taking $h = 1$

$\Rightarrow \quad 2x^3 - 3x^2 - 3x - 10 = A x(x-1)(x-2) + B x(x-1) + Cx + D$

$= A(x^3 - 3x^2 + 2x) + B(x^2 - x) + Cx + D$

comparing the coeff

$x^3 \Rightarrow \quad A = 2$

$x^2 \Rightarrow \quad -3A + B = -3 \Rightarrow -6 + B = -3 \Rightarrow B = 3$

$x \Rightarrow \quad 2A - B + C = -3 \Rightarrow 4 - 3 + C = -3 \Rightarrow C = -4$

const $\Rightarrow \quad D = -10$.

$\therefore \quad f(x) = 2x(x-1)(x-2) + 3x(x-1) - 4x - 10$

$f(x) = 2x^{(3)} + 3x^{(2)} - 4x^{(1)} - 10$

$\Delta f(x) = 6x^{(2)} + 6x^{(1)} - 4$

$\Delta^2 f(x) = 12x^{(1)} + 6$

$\Delta^3 f(x) = 12$

$\Delta^4 f(x) = 0$

IMP

* Newton forward difference Interpolation formula –

consider the set all $(n+1)$ equidistant values of the function $y = f(x)$ say $(a, f(a)), (a+h, f(a+h)), \dots, (a+nh, f(a+nh))$.

If $u = \frac{x-a}{h}$, where a is initial value of x

& h is dist. betn two consecutive x's (or interval diff)

then

$$f(a+nh) = f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \dots + \frac{u(u-1)\dots(u-(n-1))}{n!}\Delta^n f(a)$$

This is Newton - Gregory formula for forward interpolation

Remark - This for is useful for interpolating (finding) the value of f(x) near the starting point of the given set.

Remark - In some book Newton's forward difference interpolation formula is written as.

[$f(a) = y_0$ & $u = p$]

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

$$\cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}\Delta^n y_0$$

Ex. From the following data. Find the pressure for a temp. of 142°c

Temp°c x	140	150	160	170	180
Pressure $f(x)$ kgf/cm²	3.685	4.854	6.302	8.076	10.225

[142 is near to 140 ∴ use Newton's forward diff interpolation)

⇒

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4
140	3.685				
		1.169			
150	4.854		0.279		
		1.448		0.047	
160	6.302		0.326		0.002
		1.774		0.049	
170	8.076		0.375		
		2.149			
180	10.225				

Here $h = 10$, $a = 140$

& $x = 142$

$$u = \frac{x-a}{h} = \frac{142-140}{10} = \frac{2}{10} = 0.2$$

Newton's forward diff interpolation formula is

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$
$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a)$$

$$f(x) = 3.685 + (0.2)(1.169) + \frac{(0.2)(0.2-1)}{2}(0.279)$$
$$+ \frac{0.2(0.2-1)(0.2-2)}{6}(0.047)$$
$$+ \frac{0.2(0.2-1)(0.2-2)(0.2-3)}{24}(0.002)$$

$$= 3.685 + 0.23338 - 0.02223 + 0.0022 - 0.0000$$

$$f(x) = 3.8987$$

i.e. $f(142) = 3.8987$

Ex.2] The population of a town in the decimal census was given below. Estimate the population for the year 1895

Year	1891	1901	1911	1921	1931
Popuⁿ in (thousand)	46	66	81	93	101

⇒

x	$f(x)$	$\Delta f(x)$	Δ^2	Δ^3	Δ^4
1891	46				
		20			
1901	66		−5		
		15		2	
1911	81		−3		−3
		12		−1	
1921	93		−4		
		8			
1931	101				

Here $x = 1895$, $a = 1891$, $h = 10$

$$u = \frac{x-a}{h} = \frac{1895-1891}{10} = \frac{4}{10} = 0.4$$

Newton's forward diff. interpolation formula –

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$
$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a)$$

$$= 46 + (0.4)20 + \frac{0.4(0.4-1)(-5)}{2}$$
$$+ \frac{0.4(0.4-1)(0.4-2)}{6}(-2) + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{24}(-3)$$

$$= 46 + 8 + 0.6 - 0.128 + 0.1248$$

$$f(x) = 54.5968 = 54.60$$

population for the year 1895 is 54.60 thousand approximately.

Ex. From the following data. Find the number of students who obtained less than 45 marks

marks	30-40	40-50	50-60	60-70	70-80
No. of students	31	42	51	35	31

Below 50
31+42

Marks	No. of students	Δ	Δ^2	Δ^3	Δ^4
less than 40	31				
		42			
less than 50	73		9		
		51		-25	
less than 60	124		-16		37
		35		12	
less than 70	159		-4		
		31			
less than 80	190				

Here $x = 45$, $a = 40$, $h = 10$

$$\therefore u = \frac{x-a}{h} = \frac{45-40}{10} = \frac{5}{10} = 0.5$$

Newton's forward diff interpolation formula

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a)$$

$$= 31 + 0.5(42) + \frac{(0.5)(0.5-1)}{2}(9) + \frac{0.5(0.5-1)(0.5-2)}{6}(-25) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{24}(37)$$

$f(x) = 47.8671$

$\therefore$ Thus no. of students who obtained marks less than 45 are 48 approximately.

Ex. Find the cubic polynomial which is given by the following data

x	0	1	2	3
$f(x)$	1	0	1	10

$\Rightarrow$

x	$f(x)$	Δ	Δ^2	Δ^3
0	1			
		-1		
1	0		2	
		1		6
2	1		8	
		9		
3	10			

Here, $a = 0$, $h = 1$

$$u = \frac{x-a}{h} = \frac{x-0}{1} = x$$

Newton's forward diff interpolation formula -

$$f(x) = f(a) + u\,\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$

$$= 1 + x(-1) + \frac{x(x-1)}{\cancel{2}}(\cancel{2}) + \frac{x(x-1)(x-2)}{\cancel{6}}\cancel{6}$$

$$= 1 - x + x^2 - x + x(x^2 - 3x + 2)$$

$$= 1 - \cancel{x} + x^2 - \cancel{x} + x^3 - 3x^2 + \cancel{2x}$$

$$f(x) = x^3 - 2x^2 + 1$$

* Newtons Backward Diff Interpolation Formula -

consider the given set of $(n+1)$ equidistant values of the function $y=f(x)$ say

$[a, f(a)], [a+h, f(a+h)], \ldots [a+nh, f(a+nh)]$

Let $f(x)$ be a polynomial in x of degree n then

$$u = \frac{x-(a+nh)}{h}$$, where $a+nh$ is last value of x

h is interval difference

Newtons backward diff interpolation formula is given by

$$f(x) = f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a+nh) + \frac{u(u+1)(u+2)(u+3)}{3!}\nabla^4 f(a+nh) + \cdots$$

Remark - This formula is useful to the value of $f(x)$ near last value of x

Ex. The following table gives the area A of the circle for diff. diameter 'd'. Find A for $d = 105$

x d	80	85	90	95	100
$f(x)$ A	5026	5674	6362	7088	7854

$\Rightarrow$ Here 105 is near from last value of x

$\therefore$ use Newton's backward diff interpolation formula

x	$f(x)$	∇	∇^2	∇^3	∇^4
80	5026				
		648			
85	5674		40		
		688		-2	
90	6362		38		4
		726		2	
95	7088		40		
		766			
100	7854				

Here $x = 105$, $a+nh = 100$, $h = 5$

$$u = \frac{x-(a+nh)}{h} = \frac{105-100}{5} = \frac{5}{5} = 1$$

Newton's Backward diff interpolation formula is

$$f(x) = f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh)$$

$$+ \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a+nh)$$

$$+ \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f(a+nh)$$

$$= 7854 + 1(766) + \frac{1(\cancel{2})}{\cancel{2}}(40) + \frac{1(2)(3)}{6}(2)$$

$$+ \frac{1(\cancel{2})(\cancel{3})(\cancel{4})}{\cancel{24}}(4)$$

$$= 7854 + 766 + 40 + 2 + 4$$

$$f(x) = 8666$$

or Area $= 8666$ at $d = 105$

2] The no. of students who obtained mark in an examination are given in the following table

Marks	0-19	20-39	40-59	60-79	80-99
No. of students	41	62	65	50	17

Find the number of students who obtained less than 70 marks.

$\Rightarrow$

Marks less than x	no. of students $f(x)$	∇	∇^2	∇^3	∇^4
less than 20	41				
		62			
			3		
less than 40	103			-18	
		65			
			-15		0
less than 60	168			-18	
		50			
			-33		
less than 80	218				
		17			
less than 100	235				

$x = 70, \quad a+nh = 100, \quad h = 20$

$$u = \frac{x-(a+nh)}{h} = \frac{70-100}{20} = \frac{-30}{20} = -1.5$$

Newton's Backward diff interpolation formula

$$f(x) = f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh)$$

$$+ \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a+nh) + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f(a+nh)$$

$$= 235 - 1.5(17) + \frac{(-1.5)(-1.5+1)}{2}(-33)$$

$$+ \frac{(-1.5)(-1.5+1)(-1.5+2)}{6}(-18)$$

$$= 235 - 25.5 - 12.375 + 1.125$$

$f(x) = 198.25$

No. of students who obtain marks less than 70 are 198.

* Lagranges Interpolation formula -

Let $y=f(x)$ be conti & diff $(n+1)$ times in the interval (a,b). Given $(n+1)$ values $[x_0, f(x_0)]$, $[x_1, f(x_1)]$, ... $[x_n, f(x_n)]$ where the arguments $x_0, x_1, x_2, \ldots x_n$ are not necessarily equally spaced then Lagranges interpolation formula is given by

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} f(x_1)$$

$$+ \vdots$$

$$+ \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} f(x_n)$$

* For x_1, x_2, x_3, x_4 Lagranges interpolation formula

$$f(x) = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} f(x_1) + \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} f(x_2)$$

$$+ \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} f(x_3) + \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} f(x_4)$$

Remark - ① If values of x are equally spaced or not equally spaced then we can use Lagranges interpolation formula. [dist same or may not be sam]

② If values of x are equally spaced then only we can use Newtons forward & backword diff. interpolation formula. [dist betn two consec. x_i are same]

Ex. Using Lagranges interpolation formula fit a polynomial to the following data. Also find $f(3)$

	x_1	x_2	x_3	x_4
x	0	1	2	4
y	-12	0	6	12
	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$

⇒ Lagranges interpolation formula

$$f(x) = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} f(x_1) + \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} f(x_2)$$

$$+ \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} f(x_3) + \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} f(x_4)$$

$$f(x) = \frac{(x-1)(x-2)(x-4)}{(0-1)(0-2)(0-\cancel{4})} \overset{3}{(\cancel{-12})} + 0$$

$$+ \frac{(x-0)(x-1)(x-4)}{(2-0)(2-1)(2-4)} (6) + \frac{(x-0)(x-1)(x-2)}{(4-0)(4-1)(4-2)} (12)$$

$$= (x-1)\left[x^2-6x+8\right]\left(\frac{3}{2}\right) + \frac{x(x^2-5x+4)}{\cancel{2}(1)(-2)} \overset{3}{(\cancel{6})}$$

$$+ \frac{x(x^2-3x+2)}{\cancel{4}(\cancel{3})(2)} (\cancel{12})$$

$$= \frac{3}{2}(x^3-6x^2+8x-x^2+6x-8)$$

$$-\frac{3}{2}(x^3-5x^2+4x)$$

$$+\frac{1}{2}(x^3-3x^2+2x)$$

-24
15

$$= \frac{1}{2}\left[3\cancel{x^3} - 21x^2 + 42x - 24 - \cancel{3x^3} + 15x^2 - 12x + x^3 - 3x^2 + 2x\right]$$

$$f(x) = \frac{1}{2}\left[x^3 - 9x^2 + 32x - 24\right]$$

42 30

$$\therefore f(3) = \frac{1}{2}\left[27 - 81 + 96 - 24\right]$$

$$= \frac{1}{2}(18)$$

$$= 9$$

123
-105

Ex. Find $f(9)$ using Lagranges interpolation formula for the following data

	x_1	x_2	x_3	x_4	x_5
x	5	7	11	13	17
y	150	392	1452	2366	5202

$\Rightarrow$

$$f(x) = \frac{(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)} f(x_1)$$

$$+ \frac{(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)} f(x_2)$$

$$+ \frac{(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)} f(x_3)$$

$$+ \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)} f(x_4)$$

$$+ \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)} f(x_5)$$

put $x = 9$ ($\because$ find $f(9)$)

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} (150)$$

$$+ \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} (392)$$

$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} (1452)$$

$$+ \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} (2366)$$

$$+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} (5202)$$

$f(9) = 71$

Ex. 2) Find quadratic polynomial using Lagranges interpolation formula

	x_1	x_2	x_3
x	0	1	3
y	0	2	0

$$\Rightarrow f(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} f(x_1) + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} f(x_2)$$

$$+ \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$= \frac{(x-1)(x-3)}{(0-1)(0-3)}(0) + \frac{(x-0)(x-3)}{(1-0)(1-3)}(2) + 0$$

$$= \frac{x^2-3x}{(-2)}(2) = -x^2+3x$$

Reference - A textbook for S.Y.B.Sc., Numerical methods and its applications, Golden series, Nirali Prakashan.