K.T.S.P. Mandal's
Hutatma Rajguru Mahavidyalaya
Rajgurnagar-410505

Third Year B.Science

V– Sem (2019 CBCS Pattern)
As per the new syllabus

Subject- Classical Mechanics

Mr. N.D.Barne
Department of Physics
nileshbarne11@gmail.com

Topic- Central Force Field

Central Force field

- Central Force Field:

It is defined as, if the force between two particles always acts along the line Joining the particles, it is called a central force. Or "A force is always directed Towards a fixed point is known as central force".

Some important properties of central force fields**:**

If a particle moves in a central force field, then the following properties are valid.

Property 1: The path or orbit of the particle must be a plane curve, i.e. the particle moves in a plane. Property

2: The angular momentum of the particle is conserved, i.e. is constant.

Property 3: The particle moves in such a way that the position vector or radius vector drawn from O to the particle sweeps out equal areas in equal times. In other words, the time rate of change in area (i.e. the areal velocity) is constant. This is sometimes called the law of areas.

Property 4: A central force is conservative in nature.

- **The Two-Body Problem:**

Reduction of two body problem to equivalent one body problem:

This model is often referred to simply as the **two-body problem**.

In the case of only two particles, our equations of motion reduce simply to

$$m_1\ddot{\mathbf{r}}_1 = \mathbf{F}_{21}\ ;\ m_2\ddot{\mathbf{r}}_2 = \mathbf{F}_{12} \qquad (1)$$

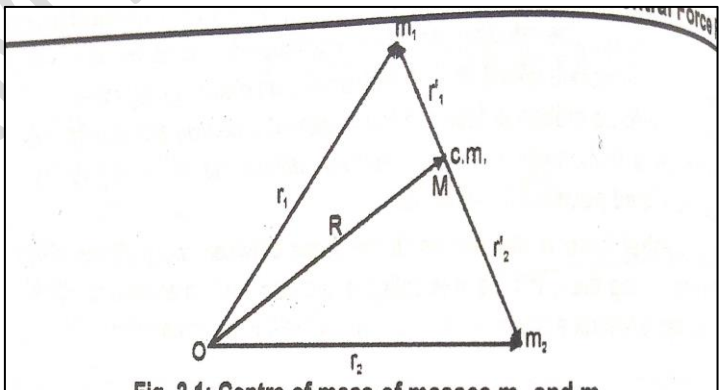

Fig. 2.1: Centre of mass of masses m_1 and m_2

$$R = m_1r_1 + m_2r_2 / m_1+m_2 \qquad 2$$

$r = r_1 - r_2$ 3

using 1 and 2 we get,

$r_1 = R + m_2 / m_1+m_2 * r$ 4

$r_2 = R - m_1/ m_1+m_2 * r$ 5

The equation of motion of the two particles individually and as a system of two particles can be written as :

$m_1\ddot{r}_1 = F_1+ F_2$ 6

$m_2\ddot{r}_2 = F_1+ F_2$ 7

Multiply equation 3 by m_2 and equation 4 by m_1 and subtracting, we get

$m_1m_2(\ddot{r}_1 - \ddot{r}_2) = (m_2 F_{12} - m_2 F_{21}) + m_1m_2 (F_1 / m_1 - F_2 / m_2) m_1m_2 \ddot{r} (m_1+m_2) F_{12} +$

$m_1m_2 \quad (F_1 / m_1 - F_2 / m_2)$ 8

Dividing equation 8 by m_1+m_2 and defining the reduced mass of the system by the formula,

$\mu = m_1m_2 / m_1 + m_2$ or $1/ \mu = 1/ m_1 + 1/ m_2$ 9

equation 8 becomes ,

$\mu\ddot{r} = F_{12} + \mu (F_1 / m_1 - F_2 / m_2)$ 10

i) If no external force , $F_1 = F_2$

ii) If external forces are proportional to masses then ,

$F_1 / m_1 = F_2 / m_2$

Equation 10 reduces to

$\mu\ddot{r} = F_{12}$ 11

The equation 11 is reduced mass equation of two body problem.

Conservation of Angular Momentum

Since our two particles interact with each other through a central potential, we know that the total angular momentum of the system is conserved. However, since we have reduced our problem to a one-particle system, it makes more sense to reformulate this statement in terms of the angular momentum of this fictitious particle,

$$\mathbf{L} = m_* \mathbf{r} \times \mathbf{v}, \tag{17}$$

where

$$\mathbf{v} = \dot{\mathbf{r}}. \tag{18}$$

Now, a short exercise in the chain rule shows us that

$$\mathbf{F}(\mathbf{r}) = -\frac{\partial}{\partial \mathbf{r}} U(|\mathbf{r}|) = -\frac{\mathbf{r}}{r}\frac{dU(r)}{dr} \quad ; \quad r = |\mathbf{r}|. \tag{19}$$

Therefore, the torque on the particle due to **F** is

$$\tau = \mathbf{r} \times \mathbf{F} \propto \mathbf{r} \times \mathbf{r} = 0. \tag{20}$$

That is, the torque vanishes because the force is parallel to the displacement vector. Thus, in the absence of any torque, the angular momentum of the particle must be constant,

$$\frac{d}{dt}\mathbf{L} = 0. \tag{21}$$

This fact is a general result for the motion of a particle in an external central potential.

For our one-particle system, conservation of angular momentum allows us to make a further simplification. For any three vectors, we can form the **scalar triple product**,

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}). \tag{22}$$

The fact that all three of these expressions are equal is left as an exercise on your homework. If we use this identity, we can see that

$$\mathbf{r} \cdot \mathbf{L} = m_* \mathbf{r} \cdot (\mathbf{r} \times \mathbf{v}) = m_* \mathbf{v} \cdot (\mathbf{r} \times \mathbf{r}) = 0. \tag{23}$$

Because this inner product is zero, it must be the case that **r** is always *perpen-dicular* to the angular momentum **L**,

$$\mathbf{r} \perp \mathbf{L} \tag{24}$$

However, because the angular momentum is constant, *there must be* *a fixed vector in space which the position vector* **r** *is always perpendicular to*. Given that the position vector is always perpendicular to a certain orientation in space, it must be the case that the position vector *always lies in a plane*.

As a result of this fact, not only has our problem been reduced to a one- particle system, it has also been effectively reduced to two dimensions. Because our problem is described by a radial force in two dimensions, at this point it is most convenient to switch over to polar coordinates,

$$r_x = r\cos\theta \ ; \ r_y = r\sin\theta. \tag{25}$$

We have chosen the convention that the plane which the particle travels in is the *x*-*y* plane, and that the angular momentum is oriented along the *z*-axis. In this set of coordinates, we can write

$$\frac{d\theta}{dt} = l/m\ r^2 \ ; \ l \equiv |\mathbf{L}|, \tag{26}$$

which you'll show on your homework. This expression for the time derivative of the angular coordinate makes another fact clear - the sign of $d\theta/dt$ is always positive, so that the particle always rotates around the center of our coordinate system in the same direction.

Conservation of Energy

There is one last conservation law we of course have at our disposal, which is the conservation of energy. Since our particle's motion is described in terms of a potential energy function, we know that the quantity.

The central force is given as ,

F (r) = n^ F (r)

Since the conservative force,

F (r) = - ∇ V(r)

V(r) is scalar function.

The curl of gradient of scalar function is always zero .

$$\vec{\nabla} \times \vec{\nabla}\, V(r) = 0$$

$$\vec{\nabla} \times F(r) = 0$$

Thus central force whose magnitude is a function of distance from the centre is a conservative one, for which the principle of conservation of total energy holds good.

- Equation of Orbit :

Recall the basic equations of motion as they will be our starting point:

$$(\ddot{r} - r\dot{\theta}^2) = f(r) \qquad (1)$$

$$(r\ddot{\theta} + 2\dot{r}\,\dot{\theta}) = 0 \qquad (2)$$

we derived the following *constant of the motion*:

$$r^2\,\dot{\theta} = h = constant \qquad (3)$$

This constant of the motion will allow you to determine the θ component of motion, provided you know the r component of motion. However, (1) and (2) are coupled (nonlinear) equations for the r and θ components of the motion. How could you solve them without solving for both the r and θ components? This is where alternative forms of the equations of motion are useful. Let us rewrite (1) in the following form (by dividing through by the mass m):

$$\ddot{r} - r\dot{\theta}^2 = f(r) \qquad (4)$$

Now, using (3), (4) can be written entirely in terms of r:

$$\ddot{r} - h^2/r^3 = f(r)/m \qquad (5)$$

We can use (5) to solve for (t), and the use (3) to solve for (t).

Equation (5) is a nonlinear differential equation. There is a useful change of variables, which for certain important central forces, turns the equation into a linear differential equation with constant coefficients, and these can always be solved analytically. Here we describe this coordinate transformation.

Let

$$r = r/u$$

This is part of the coordinate transformation. We will also use θ as a new "time" variable. Coordinate transformation are effected by the chain rule, since this allows us to express derivatives of "old" coordinates in terms of the "new" coordinates. We have

$$\dot{r} = d/dt = d/d\theta * d\theta/dt = h/r^2 * dr/d\theta = h/r^2 * d/du * du/d\theta = -h * du/d\theta \qquad (6)$$

$\ddot{r}$ =dr/dt = d/dt(-h*du/d θ) = d/ d θ (-h*du/d θ) d θ/dt = $-h^2u^2\, d^2u/d\theta^2$ (7)

Now

$$r\,\dot{\theta}^2 = rh^2/r^4 = h^2u^3 \qquad (8)$$

Substituting this relation, along with (7) into (1), gives

$$m\,(-h^2u^2 * d^2u/d\theta^2 - h^2u^3) = f\,(1/u), \qquad (9)$$

$$d^2u/d\theta^2 + u = -f\,(1/u)/mh^2u^2 \qquad (10)$$

Now , if $f(r) = k/r^2$

K is constant, equation 10 becomes a linear, constant coefficient equation .

- KEPLER'S LAWS OF PLANETARY MOTION :

  Kepler's laws of planetary motion are:

1. Planets move around the Sun in ellipses, with the Sun at one focus.
2. The line connecting the Sun to a planet sweeps equal areas in equal times.
3. The square of the orbital period of a planet is proportional to the cube of the semi major axis of the ellipse.

1. Kepler's First Law of Planetary Motion (Elliptic Orbit Law):

Statement: Each planet moves in an ellipse with the sun at one focus.

When the energy is negative, $E < 0$, and according to Equation,

$$\varepsilon = \left(1 + \frac{2EL^2}{\mu (G m_1 m_2)^2}\right)^{\frac{1}{2}} \qquad (1)$$

and the eccentricity must fall within the range $0 \le \varepsilon < 1$. These orbits are either circles or ellipses. Note the elliptic orbit law is only valid if we assume that there is only one central force acting. We are ignoring the gravitational interactions due to all the other bodies in the universe, a necessary approximation for our analytic solution.

2. Kepler's Second Law of Planetary Motion (Equal Area Law):

Statement: The radius vector from the sun to a planet sweeps out equal areas in equal time.

Using analytic geometry in the limit of small $\Delta\theta$, the sum of the areas of the triangles in Figure 9 is given by

$$\Delta A = \frac{1}{2}(r\,\Delta\theta)r + \frac{(r\,\Delta\theta)}{2}\Delta r \qquad (2)$$

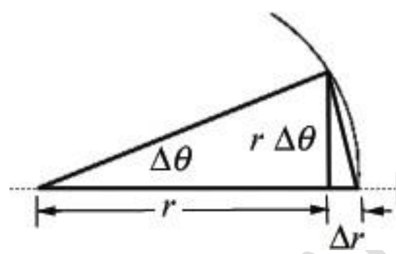

Figure 9 Kepler's equal area law.

The average rate of the change of area, ΔA, in time, Δt, is given by

$$\Delta A = \frac{1}{2}\frac{(r\,\Delta\theta)r}{\Delta t} + \frac{(r\,\Delta\theta)}{2}\frac{\Delta r}{\Delta t}. \qquad (3)$$

In the limit as $\Delta t \to 0$, $\Delta\theta \to 0$, this becomes

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}. \qquad (4)$$

Recall that according to Equation (reproduced below as Equation (5)), theangular momentum is related to the angular velocity $d\theta / dt$ by

$$d\Theta/dt = L / \mu\, r^2 \qquad (5)$$

and Equation (4) is then

$$dA/dt = L / 2\mu \qquad (6)$$

Because L and μ are constants, the rate of change of area with respect to time is a constant. This is often familiarly referred to by the expression: *equal areas are swept out in equal times* (see Kepler's Laws at the beginning of this chapter).

3. Kepler's Third Law of Planetary Motion (Period Law):

Statement: The period of revolution T of a planet about the sun is related to the semi-majoraxis a of the ellipse by $T^2 = k\,a^3$ where k is the same for all planets.

When Kepler stated his period law for planetary orbits based on observation, he only noted the dependence on the larger mass of the sun. Because the mass of the sun is much greater than the mass of the planets, his observation is an excellent approximation.

In order to demonstrate the third law we begin by rewriting Equation (6) in the form

$$2\mu \frac{dA}{dt} = L. \qquad (7)$$

Equation (7) can be integrated as

$$\int_{orbit} 2\mu\, dA = \int_0^T L\,dt\,, \qquad (8)$$

where T is the period of the orbit. For an ellipse,

$$\text{Area} = \int_{orbit} dA = \pi\, ab\,, \qquad (9)$$

where a is the semi-major axis and b is the semi-minor axis (Figure 10).

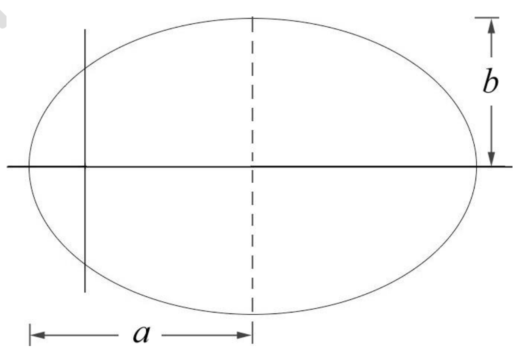

Figure 10 Semi-major and semi-minor axis for an ellipse

Thus we have

Squaring Equation (10) then yields

$$T = \frac{2\mu\pi\, ab}{L}. \qquad (10)$$

$$T^2 = \frac{4\pi^2\mu^2a^2b^2}{L^2}. \tag{11}$$

the angular momentum in terms of the semi- major axis and the eccentricity. Substitution for the angular momentum into Equation (11) yields

$$T^2 = \frac{4\pi^2\mu^2a^2b^2}{\mu G m_1 m_2 a(1-\varepsilon^2)}. \tag{12}$$

the semi-minor axis which upon substitutioninto Equation (12) yields

$$T^2 = \frac{4\pi^2\mu^2a^3}{\mu G m_1 m_2}. \tag{13}$$

Using Equation (1) for reduced mass, the square of the period of the orbit isproportional to the semi-major axis cubed,

$$T^2 = \frac{4\pi^2a^3}{G(m_1 + m_2)}. \tag{14}$$

- Artificial Satellite and its Orbit:
  In order to understand satellites and the remote sounding data obtained by instrumentslocated on satellites, we need to know something about orbital mechanics, especially the orbits in which satellites are constrained to move and the geometry with which they view the Earth.

Orbital Mechanics

The use of satellites as platforms for remote sounding is based on some veryfundamental physics.

Newton's Laws of Motion and Gravitation (1686)

→ the basis for classical mechanicsLaws of

motion:

(1) Every body continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by a force impressed upon it.

(2) The rate of change of momentum is proportional to the impressed force and is inthe same direction as that force.

Momentum = mass × velocity, so Law (2) becomes $\dot{F} = \frac{d(m\dot{v})}{dt} = m\frac{d\dot{v}}{dt} = m\dot{a}$

for constant mass

(3) For every action, there is an equal and opposite reaction.

Law of gravitation:

The force of attraction between any two particles is

- proportional to their masses
- inversely proportional to the square of the distance between them

i.e. $F = \frac{Gm_1m_2}{r^2}$ (treating the masses as points)

where

G = gravitational constant = 6.673×10^{-11} Nm^2/kg^2

These laws explain how a satellite stays in orbit.

Law (1): A satellite would tend to go off in a straight line if no force were applied to it.Law

(2): An attractive force makes the satellite deviate from a straight line and orbit Earth.

Law of Gravitation:

This attractive force is the gravitational force between Earth and the satellite.Gravity provides the inward pull that keeps the satellite in orbit.

Assuming a circular orbit, the gravitational force must equal the centripetal force.

$$\frac{mv^2}{r} = \frac{Gmm_E}{r^2}$$

where

v = tangential velocity

r = orbit radius = R_E + h (i.e. not the altitude of the orbit)

R_E = radius of Earth

h = altitude of orbit = height above Earth's surfacem =

mass of satellite

m_E = mass of Earth

$\therefore$ $v = \sqrt{\frac{Gm_E r}{}}$, so v depends only on the altitude of the orbit (not on the satellite's mass).

The period of the satellite's orbit is $T= 2\pi\,(R^3/GM)^{1/2}$

A) Geostationary orbit:

A satellite moving in a geostationary orbit remains at a fixed point in the sky at all times. This is desirable for radio communications because it allows the use of stationary antennas on the ground.

To be geostationary, the orbit must meet three criteria: -

i) The orbit must be geosynchronous. i.e. period of revolution is 24 hours .

ii) The orbit must be a circle.

iii) The orbit must lie in the earth's equatorial plane

B) Geosynchronous and Geostationary Orbits

A geosynchronous orbit is one in which the satellite orbits at the same angular velocityas the Earth. Note: geosynchronous ≠ geostationary

T = length of sidereal day = 86,164.1 s = 23 hours 56 minutes 4.1 seconds

$G = 6.67 * 10^{-11}$ N- m^2 kg

$M = 6 * 10^{24}$ kg

From Kepler's third law ,

R= 42241.2 km

The radius of the earth is , $R_e = 6370$ km

The height of geosynchronous satellite above surface of earth is ,

$R - R_e = 3581.2$ km