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Chapter 3: Numerical Differentiation and Integration

Topic- Numerical Differentiation, Trapezoidal rule, Simpson's $1/3^{rd}$ rule, Simpson's $3/8^{th}$ rule,

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Numerical Integration

* General Quadrature Formula for Equidistant Ordinates -

Let $I = \int_a^b y\,dx$ where $y = f(x)$ be a function given for equally spaced values of arguments, say $a = x_0, x_0+h, x_0+2h, \ldots, x_0+nh = b$. Let $y_0, y_1, y_2, \ldots y_n$ be the values of $f(x)$ at $x_0, \ldots, x_n$ respectively. Then General quadrature formula is given by

$$I = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \frac{1}{3!}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \cdots \text{(upto (n+1) terms)}\right.$$

By putting $n = 1, 2, 3, \ldots$ we obtain different quadrature formulae. where $h = \frac{b-a}{n}$, n is number of subintervals.

* Trapezoidal Rule -

Put $n=1$ in general quadrature formula we get

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\right]$$

* Simpson's $\frac{1}{3}$rd Rule -

Put $n=2$ in general quadrature formula we get

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \cdots + y_{n-2})\right]$$

where n is even

* Simpson's $\frac{3}{8}$th Rule -

put $n=3$ in general quadrature formula we get

$$\int_0^b f(x)\,dx = \frac{3h}{8}\Big[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+\cdots+y_{n-1}) + 2(y_3+y_6+y_9+\cdots+y_{n-3})\Big]$$

where n is multiple of 3.

Ex. 1] Evaluate $\int_0^{\pi/4} \tan x\,dx$ by Trapezoidal rule from the values provided in the following table

x	0	$\pi/8$	$\pi/4$
$\tan x$	0	0.4141	1
	y_0	y_1	y_2

here $h = \frac{\pi}{8} - 0 = \frac{\pi}{8}$

$$I = \int_0^{\pi/4} f(x)\,dx = \frac{h}{2}\left[(y_0+y_2) + 2(y_1)\right]$$

$$= \frac{\frac{\pi}{8}}{2}\left[(0+1) + 2(0.4141)\right]$$

$$= \frac{\pi}{16}\left[1.8282\right]$$

$$= 0.3590.$$

Ex. 2] A curve is drawn to pass through the point given by the following table

x	1	1.5	2	2.5	3	3.5	4
$f(x)$	2	2.4	2.7	2.8	3.0	2.6	2.1
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Estimate the area bounded by the curve, the x-axis and the lines $x=1$ & $x=4$.

$\Rightarrow$ Here $h = 1.5 - 1 = 0.5$

By simpson's $\frac{3}{8}$th rule

$$\text{Area} = \int_1^4 f(x)\,dx$$

$$I = \frac{3h}{8}\left[(y_0+y_6) + 2(y_3) + 3(y_1+y_2+y_4+y_5)\right]$$

$$= \frac{3(0.5)}{8}\left[(2+2.1) + 2(2.8) + 3(2.4+2.7+3+2.6)\right]$$

$$= 0.1875\left[4.1 + 5.6 + 32.1\right]$$

$$\text{Area} = 7.8375$$

Ex. 3] Find $\int_0^{\pi/2} \sin x \, dx$ by Trapezoidal, Simpson's $\frac{1}{3}^{rd}$, and $\frac{3}{8}^{th}$ rule from the values provided in the table.

x	0	$\frac{\pi}{12}$	$\frac{2\pi}{12}$	$3\pi/12$	$4\pi/12$	$5\pi/12$	$\pi/2$
$\sin x$	0	0.2588	0.5	0.7071	0.86603	0.9659	1
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$\Rightarrow$ 1] Trapezoidal rule

$$h = \frac{\pi}{12}$$

$$I = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{\pi}{12} \times \frac{1}{2}\left[(0+1) + 2(0.2588 + 0.5 + 0.7071 + 0.86603 + 0.9659)\right]$$

$$= \frac{\pi}{24}\left[7.59566\right]$$

$$\int_0^{\pi/2} \sin x \, dx = 0.9943$$

2] Simpson's $\frac{1}{3}^{rd}$ rule

$$I = \frac{h}{3}\left[(y_0+y_6) + 2(y_2+y_4) + 4(y_1+y_3+y_5)\right]$$

$$= \frac{\pi}{12} \times \frac{1}{3}\left[(0+1) + 2(0.5 + 0.86603) + 4(0.2588 + 0.7071 + 0.9659)\right]$$

$= \frac{\pi}{36}[1 + 2.7321 + 7.7272]$

$= \frac{\pi}{36}[11.4593]$

$\int_0^{\pi/2} \sin x \, dx = 1.00001$

3] Simpson's $\frac{3}{8}^{th}$ rule -

$I = \frac{3h}{8}[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)]$

$= \frac{3\left(\frac{\pi}{12}\right)}{8}[(0+1) + 2(0.7071) + 3(0.2588 + 0.5 + 0.86603 + 0.9659)]$

$= \frac{\pi}{32}[1 + 1.4142 + 7.7722]$

$I = 1.00005$

Ex. 4] By using Simpson's $\frac{1}{3}^{rd}$ rule and find the approximate value of $\int_0^{\pi/2} \sqrt{\cos x}\, dx$.

$\Rightarrow$ Take $n = 8$, $a = 0$, $b = \frac{\pi}{2}$

$h = \frac{b-a}{n}$

$h = \frac{\frac{\pi}{2} - 0}{8} = \frac{\pi}{16}$

x	0	$\pi/16$	$2\pi/16$	$3\pi/16$	$4\pi/16$	$5\pi/16$	$6\pi/16$	$7\pi/16$
$\sqrt{\cos x}$	1	0.9903	0.9612	0.9118	0.8409	0.7456	0.6186	0.4416
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7

x	$\pi/2$
$\sqrt{\cos x}$	0
	y_8

Simpson's $\frac{1}{3}$rd rule

$$\int_0^{\pi/2} \sqrt{\cos x}\, dx = \frac{h}{3}\left[(y_0+y_8) + 2(y_2+y_4+y_6) + 4(y_1+y_3+y_5+y_7)\right]$$

$$= \frac{\pi}{16} \times \frac{1}{3}\left[(1+0) + 2(0.9612 + 0.8409 + 0.6186) + 4(0.9903 + 0.9118 + 0.7456 + 0.4416)\right]$$

$$= \frac{\pi}{48}\left[1 + 4.8414 + 12.3572\right]$$

$$\int_0^{\pi/2} \sqrt{\cos x}\, dx = 1.1911$$

Ex. 5] Compute the value of $\log 2$ from the formula $\log 2 = \int_1^2 \frac{1}{x}\, dx$ using Simpson's rule taking 10 subintervals.

$\Rightarrow$ Here $a = 1$, $b = 2$ & $n = 10$

$$\therefore h = \frac{b-a}{n} = \frac{2-1}{10} = \frac{1}{10} = 0.1$$

x	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7
$\frac{1}{x}$	1	0.9090	0.8333	0.7692	0.7142	0.6667	0.625	0.5882
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7

x	1.8	1.9	2
$\frac{1}{x}$	0.5556	0.5263	0.5
	y_8	y_9	y_{10}

By Simpson's $\frac{1}{3}$rd rule

$$\int_1^2 \frac{1}{x}\, dx = \frac{h}{3}\left[(y_0+y_{10}) + 2(y_2+y_4+y_6+y_8) + 4(y_1+y_3+y_5+y_7+y_9)\right]$$

$$= \frac{0.1}{3}\left[1.5 + 5.4562 + 13.8376\right]$$

$$\int_1^2 \frac{1}{x}\, dx = 0.6931 \quad \text{—}(1)$$

Now, $\int_1^2 \frac{1}{x} dx = [\log x]_{x=1}^{2}$

$= \log 2 - \log 1$

$= \log 2 - 0$

$= \log 2 \quad \text{—} (2)$

From (1) & (2)

$\therefore \log 2 = 0.6931$

Example-6] Estimate $\int_1^5 \log_e x$ dividing the interval into four equal parts and compare with correct value.

$\Rightarrow$ Take $n=4$, $a=1$, $b=5$

$\therefore h = \frac{b-a}{n} = \frac{5-1}{4} = 1$

$\log_e x = \ln$

x	1	2	3	4	5
$\log_e x$	0	0.6931	1.0986	1.3863	1.6094
	y_0	y_1	y_2	y_3	y_4

By Simpson's $\frac{1}{3}^{rd}$ rule

$$\int_1^5 \log_e x\, dx = \frac{h}{3}\left[(y_0+y_4) + 2(y_2) + 4(y_1+y_3)\right]$$

$$= \frac{1}{3}\left[1.6094 + 2.1972 + 8.3176\right]$$

$$= 4.0414$$

Now

$$\int_1^5 \log_e x\, dx = \int_1^5 \log x \cdot 1\, dx$$

$$= \left[\log x\,(x) - \int \frac{1}{x}\cdot x\, dx\right]_1^5$$

$$= \left[x \log x - x\right]_1^5$$

$$= [5\log 5 - 5] - [0-1]$$

$$= 4.0472$$

$$\text{Error} = |4.0472 - 4.0414|$$
$$= 0.0058.$$

Example 7 – Estimate the value of $\int_2^{10} \frac{dx}{1+x}$ by using Simpson's $\frac{1}{3}^{rd}$ rule (take $h=1$) also find error.

Sol^n – Here $h = 1$, $a = 2$, $b = 10$

$$f(x) = \frac{1}{1+x}$$

x	2	3	4	5	6	7	8	9	10
$\frac{1}{1+x}$	0.3333	0.25	0.2	0.1667	0.1428	0.125	0.1111	0.1	0.0909
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

By Simpson's $\frac{1}{3}^{rd}$ rule

$$\int_2^{10} \frac{1}{1+x}\,dx = \frac{h}{3}\left[(y_0+y_8) + 2(y_2+y_4+y_6) + 4(y_1+y_3+y_5+y_7)\right]$$

$$= \frac{1}{3}\left[0.4242 + 0.9078 + 2.5668\right]$$

$$= 1.2996$$

Now
$$\int_2^{8} \frac{1}{1+x}\,dx = \left[\ln(1+x)\right]_2^8$$

$$= \ln(9) - \ln(3)$$
$$= 2.1972 - 1.0986$$
$$= 1.0986$$

$$\text{Error} = |1.2996 - 1.0986|$$
$$= 0.201$$

Ex. 8] By using Simpson's $\frac{1}{3}^{rd}$ rule find the area of the cross-section of a river 80 meters wide, the depth y (in meters) at a distance x

from the bank is given in the table below

x	0	10	20	30	40	50	60	70	80
y	0	4	7	9	12	15	14	8	3
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

$\Rightarrow$ $$\text{Area} = \int_0^{80} f(x)\,dx \quad , \quad h = 10$$

By Simpson's $\frac{1}{3}^{rd}$ rule

$$\int_0^{80} f(x)\,dx = \frac{h}{3}\left[(y_0+y_8) + 2(y_2+y_4+y_6) + 4(y_1+y_3+y_5+y_7)\right]$$

$$= \frac{10}{3}\left[3 + 66 + 144\right]$$

$$\text{Area} = 710 \text{ sq. mt.}$$

Ex. 9 – The velocity of a vehicle (running on a straight road) at interval of 2 minutes are given below

Time in minutes	0	2	4	6	8	10	12
Velocity in km/hr	0	22	30	27	18	7	0

Apply Simpson's rule to obtain the distance covered by the vehicle.

$\Rightarrow$

$$\text{distance} = \int \text{velocity}$$

Time in minutes	0	2	4	6	8	10	12
vel. in km/min	0	22/60	30/60	27/60	18/60	7/60	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$h = 2$$

By Simpson's $\frac{3}{8}^{th}$ rule

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0+y_6) + 2(y_3) + 3(y_1+y_2+y_4+y_5)\right]$$

$$\doteq \frac{3(2)}{8}\left[0+2\left(\frac{27}{60}\right)+3\left(\frac{22}{60}+\frac{30}{60}+\frac{18}{60}+\frac{7}{60}\right)\right]$$

$$= \frac{6}{8}\times\frac{1}{60}\left[54+231\right]$$

$$= \frac{1}{80}\left[285\right]$$

$$= 3.5625$$

$\therefore$ dist = 3.5625 km

Ex. Estimate the area bdd by the curve, the x-axis, and the extreme ordinates, a curve is given by the points (x, y) given below
(0, 23), (0.5, 19), (1, 14), (1.5, 11), (2, 12.5), (2.5, 13), (3, 19), (3.5, 20), (4, 20).

$\Rightarrow$

x	0	0.5	1	1.5	2	2.5	3	3.5	4
y	23	19	14	11	12.5	13	19	20	20
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

$h = 0.5$

By Simpson's $\frac{1}{3}$rd rule

$$\text{Area} = \int_0^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_8)+2(y_2+y_4+y_6)+4(y_1+y_3+y_5+y_7)\right]$$

$$= \frac{0.5}{3}\left[46+91+252\right]$$

$$= 64.8333 \text{ sq. units.}$$

Refrences - 1) Text book of S.Y.B.Sc. Numerical methods and its application by Nirali Prakashan.
2) Text book of S.Y.B.Sc. Numerical methods and its application, Golden series by Nirali Prakashan.

Chapter - 3
Numerical Differentiation and Integration.

* Numerical differentiation -

1) If we have to find $\frac{dy}{dx}$ at the point x_0, near to intial value of table we use

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

2) If we have to find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ at the point x_n, near to end point, we use

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \cdots\right]$$

Ex. Find $\frac{dy}{dx}$ at $x = 0.6$ & $\frac{d^2y}{dx^2}$ at $x = 0.4$ from the following data

x	0.4	0.5	0.6	0.7	0.8
y	1.5836	1.7974	2.0442	2.3275	2.6511

$\Rightarrow$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.4	1.5836				
		0.2138			
0.5	1.7974		0.033		
		0.2468		0.0035	
0.6	2.0442		0.0365		0.0003
		0.2833		0.0038	
0.7	2.3275		0.0403		
		0.3236			
0.8	2.6511				

$x_0 = 0.6 \quad , \quad h = 0.1$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$= \frac{1}{0.1}\left[0.2833 - \frac{1}{2}(0.0403)\right]$$

$$= \frac{1}{0.1}\left[0.26315\right] = 2.6315$$

2) $\frac{d^2y}{dx^2}$ at $x = 0.4$, $h = 0.1$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

$$= \frac{1}{(0.1)^2}\left[0.033 - 0.0035 + \frac{11}{12}(0.0003)\right]$$

$$= \frac{1}{(0.1)^2}\left[-0.002 + 0.000275\right]$$

$$= \frac{1}{(0.1)^2}\left[-0.001725\right]$$

$$= -0.1725$$

2] Find $\frac{dy}{dx}$ at $x = 1$ from the following table

x	1	2	3	4
y	1	8	27	64

$\Rightarrow$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	1			
		7		
2	8		12	
		19		6
3	27		18	
		37		
4	64			

$x_0 = 1, \quad h = 1$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \cdots\right]$$

$$= \frac{1}{1}\left[7 - \frac{1}{2}(12) + \frac{1}{3}(6)\right]$$

$$= [7 - 6 + 2]$$

$$= 3$$

3] A rod rotating in a plane about one of its end. The angle θ in radian at a different times t in seconds are given below

x / t	0	0.2	0.4	0.6	0.8	1
y / θ	0	0.15	0.5	1.15	2	3.20

Find its angular velocity and angular acceleration when $t = 0.6$

$\Rightarrow$ angular velocity $= \dfrac{d\theta}{dt} = \dfrac{dy}{dx}$

angular acceleration $= \dfrac{d^2\theta}{dt^2} = \dfrac{d^2y}{dx^2}$

i.e. to find $\dfrac{dy}{dx}$ & $\dfrac{d^2y}{dx^2}$ at $x = 0.6$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
0	0					
		0.15				
			0.2			
0.2	0.15			0.1		
		0.35				
					-0.2	
0.4	0.5		0.3			0.45
				-0.1		
		0.65				
					0.25	
0.6	1.15		0.2			
		0.85		0.15		
			0.35			
0.8	2					
		1.2				
1	3.2					

$x_n = 0.6, \quad h = 0.2$

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n\right]$$

$$= \frac{1}{0.2}\left[0.65 + \frac{1}{2}(0.3) + \frac{1}{3}(0.1)\right]$$

$$= \frac{1}{0.2}\left[0.65 + 0.15 + 0.0333\right]$$

$$= 4.1665$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n\right]$$

$$= \frac{1}{(0.2)^2}\left[0.3 + 0.1\right]$$

$$= \frac{1}{(0.2)^2}(0.4)$$

$$= 10$$

Ex. Find $\frac{dy}{dx}$ at $x = 0.2$ & $\frac{d^2y}{dx^2}$ at $x = 0$

x	0	0.2	0.4	0.6	0.8	1
y	1	1.16	3.56	13.96	41.96	101.0

$\Rightarrow$

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	1					
		0.16				
0.2	1.16		2.24			
		2.4		5.76		
0.4	3.56		8		3.84	
		10.4		9.6		0
0.6	13.96		17.6		3.84	
		28		13.44		
0.8	41.96		31.04			
		59.04				
1	101.00					

1) $\frac{dy}{dx}$ at $x = 0.2$, Here $x_0 = 0.2$, $h = 0.2$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0\right]$$

$$= \frac{1}{0.2}\left[2.4 - \frac{1}{2}(8) + \frac{1}{3}(9.6) - \frac{1}{4}(3.84)\right]$$

$$= 3.2$$

2) $\frac{d^2y}{dx^2}$ at $x = 0$, Take $x_0 = 0$, $h = 0.2$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right]$$

$$= \frac{1}{(0.2)^2}\left[2.24 - 5.76 + \frac{11}{12}(3.84)\right]$$

$$= \frac{1}{(0.2)^2}\left[-3.52 + 3.52\right] = 0$$

Ex. Find the first derivative of y at $x = 0.4$ from the following table

x	0.1	0.2	0.3	0.4
y	1.10517	1.22140	1.34986	1.49182

$\Rightarrow$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
0.1	1.10517			
		0.11623		
0.2	1.22140		0.01223	
		0.12846		0.00127
0.3	1.34986		0.01350	
		0.14196		
0.4	1.49182			

Here $x_n = 0.4$ & $h = 0.1$

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \cdots\right]$$

$$= \frac{1}{0.1}\left[0.14196 + \frac{1}{2}(0.0135) + \frac{1}{3}(0.00127)\right]$$

$$= \frac{1}{0.1}\left[0.14196 + 0.00675 + 0.0004\right]$$

$$= 1.4911$$

Reference: Numerical methods and its applications, text book for S.Y.B.Sc., by Golden series, Nirali Prakashan