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Chapter 4: Numerical solution of first order ordinary differential equations

Topic- Taylor's Series method, Picard's method of successive approximations, Euler's method, Modified Euler's methods, Runge - Kutta Methods

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* Runge Kutta Method of fourth order –

consider a diff eqn

$$\frac{dy}{dx} = f(x,y)$$

with initial condition

$$y(x_0) = y_0$$

then Runge-kutta formula for 4^{th} order is

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

where

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$$

$$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Remark – ① Runge-kutta method of 4^{th} order is more accurate than other methods.

② Modified Euler method is a special case of second order Runge-kutta method.

* Runge-kutta formula for finding y_2

$$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

where

$$k_1 = hf(x_1, y_1)$$

$$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$$

$$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2})$$

$$k_4 = hf(x_1 + h, y_1 + k_3).$$

Ex. 1] Use Runge-kutta fourth order formula to find $y(0.1)$, $y(0.2)$ for given initial value problem $\frac{dy}{dx} = x + y^2$, $y(0) = 1$, with $h = 0.1$

$\Rightarrow$ $f(x,y) = x + y^2$, $x_0 = 0$, $y_0 = 1$, $h = 0.1$

$x_0 = 0$, $x_1 = x_0 + h = 0.1$, $x_2 = x_1 + h = 0.2$

$y(0.1) = y_1$ & $y(0.2) = y(x_2) = y_2$

1] Find y_1

$$k_1 = h\, f(x_0, y_0)$$
$$= 0.1\ f(0, 1)$$
$$= 0.1\ [0 + 1^2]$$
$$= 0.1$$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= 0.1\ f(0 + 0.05, 1 + 0.05)$$
$$= 0.1\ f(0.05, 1.05)$$
$$= 0.1\ [0.05 + 1.05^2]$$
$$= 0.1152$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= 0.1\ f(0.05, 1 + 0.0576)$$
$$= 0.1\ [0.05 + 1.0576^2]$$
$$= 0.1168$$

$$k_4 = h\, f(x_0 + h, y_0 + k_3)$$
$$= 0.1\ f(0 + 0.1, 1 + 0.1168)$$
$$= 0.1\ f(0.1, 1.1168)$$
$$= 0.1\ [0.1 + 1.1168^2]$$
$$= 0.1347$$

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$
$$= 1 + \frac{1}{6}[0.6987]$$
$$= 1.11645$$

2) Find y_2

$$k_1 = h f(x_1, y_1)$$

$$= 0.1 f(0.1, 1.11645)$$

$$= 0.1 [0.1 + 1.11645^2]$$

$$= 0.1346$$

$$k_2 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$$

$$= 0.1 f(0.1 + 0.05, 1.11645 + 0.0673)$$

$$= 0.1 f(0.15, 1.18375)$$

$$= 0.1 [0.15 + 1.18375^2]$$

$$= 0.1 [0.15 + 1.4013]$$

$$k_2 = 0.1551$$

$$k_3 = h f(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2})$$

$$= 0.1 f(0.15, 1.11645 + 0.0776)$$

$$= 0.1 f(0.15, 1.1940)$$

$$= 0.1 [0.15 + 1.1940^2]$$

$$= 0.1576$$

$$k_4 = h f(x_1 + h, y_1 + k_3)$$

$$= 0.1 f(0.1 + 0.1, 1.11645 + 0.1576)$$

$$= 0.1 f(0.2, 1.2740)$$

$$= 0.1 [0.2 + 1.2740^2]$$

$$= 0.1823$$

$$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 1.11645 + \frac{1}{6}[0.9423] = 1.2735.$$

2] $\frac{dy}{dx} = 3x + \frac{y}{2}$, $y(0) = 1$, find $y(0.2)$ by using Runge-Kutta method of 4^{th} order.

$\Rightarrow$ $f(x,y) = 3x + \frac{y}{2}$, $x_0 = 0, y_0 = 1$, take $h = 0.2$

$x_0 = 0$, $x_1 = 0 + h = 0.2$

$y(0.2) = y(x_1) = y_1$

$k_1 = h f(x_0, y_0)$

$= 0.2 f(0, 1)$

$= 0.2 [0 + 0.5)$

$= 0.1$

$k_2 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$= 0.2 f(0 + 0.1, 1 + 0.05)$

$= 0.2 f(0.1, 1.05)$

$= 0.2 [0.3 + 0.525]$

$= 0.165$

$k_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$

$= 0.2 f(0.1, 1 + 0.0825)$

$= 0.2 [0.3 + 0.5412]$

$= 0.1682$

$k_4 = h f(x_0 + h, y_0 + k_3)$

$= 0.2 f(0 + 0.2, 1 + 0.1682)$

$= 0.2 f(0.2, 1.1682)$

$= 0.2 [0.6 + 0.5841]$

$= 0.2368$

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 1 + \frac{1}{6}[0.1 + 2(0.165) + 2(0.1682) + 0.2368]$$

$$= 1 + \frac{1}{6}[0.1 + 0.33 + 0.3364 + 0.2368]$$

$$= 1 + \frac{1}{6}[1.0032]$$

$$y_1 = 1.1672$$

i.e. $y(0.2) = 1.1672$

3] $\frac{dy}{dx} = \frac{3x+y}{x+2y}$, $y(1)=1$, find $y(1.2)$ correct upto four decimal places.

$\Rightarrow$ $f(x,y) = \frac{3x+y}{x+2y}$

Take $h = 0.2$, $y(1) = 1$

$\Rightarrow$ $x_0 = 1$, $y_0 = 1$, $x_1 = x_0 + h = 1.2$

Find y_1

$$k_1 = h f(x_0, y_0)$$

$$= 0.2 f(1,1)$$

$$= 0.2\left[\frac{3+1}{1+2}\right]$$

$$= 0.2667$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.2 f(1 + 0.1, 1 + 0.1334)$$

$$= 0.2 f(1.1, 1.1334)$$

$$= 0.2\left[\frac{3.3 + 1.1334}{1.1 + 2.2668}\right] = 0.2\left[\frac{4.4334}{3.3668}\right]$$

$k_2 = 0.2634$

$k_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$

$= 0.2 f(1.1, 1.1318)$

$= 0.2 \left[\frac{3.3 + 1.1318}{1.1 + 2.2636} \right]$

$= 0.2 \left[\frac{4.4318}{3.3636} \right]$

$= 0.2635$

$k_4 = h f(x_0 + h, y_0 + k_3)$

$= 0.2 f(1.2, 1.2635)$

$= 0.2 \left[\frac{3.6 + 1.2635}{1.2 + 2.527} \right]$

$= 0.2 \left[\frac{4.8635}{3.727} \right]$

$k_4 = 0.261$

$y_1 = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$

$= 1 + \frac{1}{6} [1.5815]$

$= 1.2636$

Reference - A text book for S.Y.B.Sc., Numerical methods and its applications, Golden series by Nirali Prakashan.

* Runge-Kutta Method of second-

consider a diff eqn with initial condition

$$\frac{dy}{dx} = f(x,y) \quad , \quad y(x_0) = y_0$$

$x \Rightarrow h$

$y \Rightarrow k$

$$k_1 = h f(x_0, y_0)$$

$$k_2 = h f(x_0+h, y_0+k_1)$$ then Runge-Kutta 2nd order formula is

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

Ex. Use Runge-Kutta's second order formula to find $y(0.2)$ & $y(0.4)$ taking $h=0.2$. Given that $y(0)=0$ and $\frac{dy}{dx} = 1+y^2$.

Let

$\Rightarrow$ $f(x,y) = 1+y^2$, $x_0 = 0$, $y_0 = 0$, $h = 0.2$

$x_0 = 0$, $x_1 = x_0 + h = 0.2$, $x_2 = x_1 + h = 0.4$

[$y(0.2) = y(x_1) = y_1$ & $y(0.4) = y(x_2) = y_2$]

1] Find y_1

$$k_1 = h f(x_0, y_0)$$
$$= 0.2 \, f(0,0)$$
$$= 0.2(1+0)$$
$$= 0.2$$

$y_0 + k_1 = 0 + 0.2 = 0.2$

$$k_2 = h f(x_0+h, y_0+k_1)$$
$$= 0.2 f(0.2, 0.2)$$
$$= 0.2(1+0.04)$$
$$= 0.208$$

$$y_1 = y_0 + \frac{1}{2}(k_1+k_2)$$
$$= 0 + \frac{1}{2}(0.2+0.208)$$
$$= 0.204$$

2) Find y_2

$k_1 = h f(x_1, y_1)$

$k_2 = h f(x_1 + h, y_1 + k_1)$

$y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$

[For y_3,

$k_1 = h f(x_2, y_2)$

$k_2 = h f(x_2 + h, y_2 + k_1)$

$y_3 = y_2 + \frac{1}{2}(k_1 + k_2)$]

$$
\begin{aligned}
k_1 &= h f(x_1, y_1) \\
&= 0.2\, f(0.2, 0.204) \\
&= 0.2\,[1 + 0.0416] \\
&= 0.2083
\end{aligned}
$$

$$
\begin{aligned}
k_2 &= h f(x_1 + h, y_1 + k_1) \\
&= 0.2\, f(0.4, 0.4123) \\
&= 0.2\,[1 + (0.4123)^2] \\
&= 0.2340
\end{aligned}
$$

$$
\begin{aligned}
y_2 &= y_1 + \frac{1}{2}(k_1 + k_2) \\
&= 0.204 + \frac{1}{2}(0.2083 + 0.2340) \\
&= 0.4252
\end{aligned}
$$

Ex. Use Runge kutta method of the second order to solve $\frac{dy}{dx} = 2(1 + x - y)$ with $y(2) = 5$, find $y(2.2)$ taking $h = 0.2$.

$\Rightarrow$ $f(x, y) = 2(1 + x - y)$, $x_0 = 2$, $y_0 = 5$, $h = 0.2$

$x_0 = 2$, $x_1 = x_0 + h = 2.2$

$[y(2.2) = y(x_1) = y_1]$

$k_1 = h f(x_0, y_0)$

$k_1 = 0.2\, f(2, 5)$

$= 0.2\,[2(1+2-5)]$

$= 0.2(-4)$

$= -0.8$

$y_0 + k_1 = 5 - 0.8$

$= 4.2$

$k_2 = h f(x_0+h, y_0+k_1)$

$= 0.2\, f(2.2, 4.2)$

$= 0.2\,[2(1+2.2-4.2)]$

$= -0.4$

$y_1 = y_0 + \frac{1}{2}(k_1+k_2)$

$= 5 + \frac{1}{2}(-0.8-0.4)$

$= 5 - 0.6$

$= 4.4$

Ex. $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$, $y(0)=1$, find $y(0.2)$ & $y(0.4)$ by using Runge-kutta method of second order.

$\Rightarrow$ $f(x,y) = \frac{y^2-x^2}{y^2+x^2}$, $x_0=0$, $y_0=1$,

Take $h=0.2$

$x_0=0$, $x_1=0.2$, $x_2=0.4$

Find $y_1 \Rightarrow$

$k_1 = h\, f(x_0, y_0)$

$= 0.2\, f(0, 1)$

$= 0.2\left[\frac{1-0}{1+0}\right]$

$k_1 = 0.2$

$k_2 = h f(x_0+h, y_0+k_1)$

$= 0.2\, f(0.2, 1.2)$

$= 0.2\left[\frac{(1.2)^2-(0.2)^2}{(1.2)^2+(0.2)^2}\right]$

$$= 0.2 \left[\frac{1.44 - 0.04}{1.44 + 0.04} \right]$$

$$= 0.2 \left[\frac{1.4}{1.48} \right]$$

$$k_2 = 0.1892$$

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$= 1 + \frac{1}{2}(0.2 + 0.1892)$$

$$= 1.1946$$

2) Find y_2

$$k_1 = h\, f(x_1, y_1)$$

$$= 0.2\, f(0.2, 1.1946)$$

$$= 0.2 \left[\frac{1.4271 - 0.04}{1.4271 + 0.04} \right]$$

$$= 0.2 \left[\frac{1.3871}{1.4671} \right]$$

$$= 0.1891$$

$$k_2 = h\, f(x+h, y_1 + k_1)$$

$$= 0.2\, f(0.4, 1.3837)$$

$$= 0.2 \left[\frac{1.9146 - 0.16}{1.9146 + 0.16} \right]$$

$$= 0.2 \left[\frac{1.7546}{2.0746} \right]$$

$$= 0.1692$$

$$y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$$

$$= 1.1946 + \frac{1}{2}(0.1891 + 0.1692)$$

$$= 1.3738.$$

* Modified Euler's method -

consider the diff eqn

$$\frac{dy}{dx} = f(x,y)$$

$y_1 = y_0 + h f(x_0, y_0)$

$y_2 = y_1 + h f(x_1, y_1)$

with initial condition

$$y(x_0) = y_0$$

Then modified Euler's iteration formulae are

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$\vdots$$

$$y_1^{(n)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n-1)})\right]$$

Take $y_1 = y_1^{(n)}$

Now,

$$y_2^{(0)} = y_1 + h f(x_1, y_1)$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

$$\vdots$$

$$y_2^{(n)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(n-1)})\right]$$

Take $y_2 = y_2^{(n)}$.

Remark - Modified Euler's method is more accurate than Euler's method.

Ex. Solve the following initial value problem using modified Euler's method

1] $\frac{dy}{dx} = 2xy$, $y(0) = 1$, at the point $x = 0.1$

Take $h = 0.1$

$\Rightarrow \quad f(x,y) = 2xy, \quad x_0 = 0, \; y_0 = 1, \; h = 0.1$

$x_0 = 0, \quad x_1 = x_0 + h = 0.1$

$y_0 \qquad y_1$

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$
$$= 1 + 0.1 \, f(0,1)$$
$$= 1 + 0.1(0)$$
$$= 1$$

$$y_1^{(0)} = 1$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(0)})\right]$$
$$= 1 + \frac{0.1}{2}\left[f(0,1) + f(0.1,1)\right]$$
$$= 1 + 0.05\left[0 + 2(0.1)(1)\right]$$
$$= 1 + 0.05\,(0.2)$$
$$= 1 + 0.01$$
$$= 1.01$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(1)})\right]$$
$$= 1 + \frac{0.1}{2}\left[0 + f(0.1, 1.01)\right]$$
$$= 1 + 0.05\left[2(0.1)(1.01)\right]$$
$$= 1.0101$$

Take $y_1 = 1.0101$.

Ex 2] $yy' = x$, $y(0) = 1.5$, find $y(0.2)$, take $h = 0.1$

$\Rightarrow \quad y' = \dfrac{x}{y}$

$f(x,y) = \dfrac{x}{y}, \quad x_0 = 0, \; y_0 = 1.5, \; h = 0.1$

$x_0 = 0, \quad x_1 = x_0 + h = 0.1, \quad x_2 = x_1 + h = 0.2$

$y_0 \qquad y_1 \qquad y_2$

$$y_1^{(0)} = y_0 + h\, f(x_0, y_0)$$

$$= 1.5 + 0.1\, f(0, 1.5)$$

$$= 1.5 + 0.1 \left(\frac{0}{1.5}\right)$$

$$= 1.5$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1.5 + \frac{0.1}{2}\left[f(0, 1.5) + f(0.1, 1.5)\right]$$

$$= 1.5 + 0.05\left[0 + \frac{0.1}{1.5}\right]$$

$$= 1.5 + 0.0033$$

$$y_1^{(1)} = 1.5033.$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1.5 + \frac{0.1}{2}\left[0 + f(0.1, 1.5033)\right]$$

$$= 1.5 + 0.05\left[\frac{0.1}{1.5033}\right]$$

$$y_1^{(2)} = 1.5033$$

Take $y_1 = 1.5033$

$$y_2^{(0)} = y_1 + h\, f(x_1, y_1)$$

$$= 1.5033 + 0.1\, f(0.1, 1.5033)$$

$$= 1.5033 + 0.1\left[\frac{0.1}{1.5033}\right]$$

$$= 1.5100$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

$$= 1.5033 + \frac{0.1}{2}\left[f(0.1, 1.5033) + f(0.2, 1.51)\right]$$

$$= 1.5033 + 0.05\left[\frac{0.1}{1.5033} + \frac{0.2}{1.51}\right]$$

$$= 1.5033 + 0.05\left[0.0665 + 0.1324\right]$$

$$= 1.5132$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

$$= 1.5033 + 0.05\left[0.0665 + f(0.2, 1.5132)\right]$$

$$= 1.5033 + 0.05\left[0.0665 + 0.1322\right]$$

$$= 1.5132$$

Take $y_2 = 1.5132$

Ex. 3] $\frac{dy}{dx} = 1 + \ln(x+y)$, $y(0) = 1$, at $x = 0.2$, take $h = 0.1$

$\Rightarrow$ $f(x,y) = 1 + \ln(x+y)$, $x_0 = 0$, $y_0 = 1$, $h = 0.1$

$x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$

y_0 $\quad$ y_1 $\quad$ y_2

$$y_1^{(0)} = y_0 + h\, f(x_0, y_0)$$

$$= 1 + 0.1\ f(0, 1)$$

$$= 1 + 0.1\left[1 + \ln(1)\right]$$

$$= 1 + 0.1\left[1 + 0\right]$$

$$= 1 + 0.1$$

$$y_1^{(0)} = 1.1$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$f(x,y) = 1 + \ln(x+y)$

$$= 1 + \frac{0.1}{2}\left[f(0,1) + f(0.1, 1.1)\right]$$

$$= 1 + 0.05\left[1 + 1 + \ln(1.2)\right]$$

$$y_1^{(1)} = 1.1091$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1 + 0.05\left[1 + f(0.1, 1.1091)\right]$$

$$= 1 + 0.05\left[1 + 1 + \ln(0.1 + 1.1091)\right]$$

$$= 1.1095$$

Take $y_1 = 1.1095$

$$y_2^{(0)} = y_1 + h f(x_1, y_1)$$

$$= 1.1095 + 0.1 \, f(0.1, 1.1095)$$

$$= 1.1095 + 0.1\left[1 + \ln(0.1 + 1.1095)\right]$$

$$= 1.1095 + 0.1\left[1.1902\right]$$

$$= 1.22285$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(0)})\right]$$

$$= 1.1095 + 0.05\left[1.1902 + f(0.2, 1.22285)\right]$$

$$= 1.1095 + 0.05\left[1.1902 + 1.3566\right]$$

$$= 1.2368$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2, y_2^{(1)})\right]$$

$$= 1.1095 + 0.05\left[1.1902 + f(0.2, 1.2368)\right]$$

$$= 1.2371$$

Take $y_2 = 1.2371$

* Euler's method for successive approximation –

consider the diff. eqn

$$\frac{dy}{dx} = f(x,y) \quad \text{with initial condition}$$

$$y(x_0) = y_0$$

$$y_1 = y_0 + h f(x_0, y_0)$$

$$y_2 = y_1 + h f(x_1, y_1)$$

$$y_3 = y_2 + h f(x_2, y_2)$$

$$\vdots$$

$$y_{n+1} = y_n + h f(x_n, y_n) \quad , \quad n = 0, 1, 2, \ldots$$

Remark – To obtain the solution with desired accuracy, we have to take smaller value of h, hence the solution is obtained very slowly. Due to this the method is rarely used. The more accurate result will be obtained by the modified Euler's method.

Ex. Solve the following initial value problem using Euler's method for value of y at the given point of x.

1) $\frac{dy}{dx} = 1 - y$, $y(0) = 0$, at the point $x = 0.2$ $(h = 0.1)$

x_0 y_0

$\Rightarrow \frac{dy}{dx} = 1 - y$

$\therefore f(x,y) = 1 - y$, $x_0 = 0$ & $y_0 = 0$

$x_0 = 0$, $x_1 = 0 + 0.1 = 0.1$, $x_2 = x_1 + h = 0.1 + 0.1$
$= 0.2$

x_0, x_1, x_2

y_0, y_1, y_2

	x_1	x_2	x_3
x	0 ,	0.1 ,	0.2
y	y_0	y_1	y_2

$y_1 = y_0 + h f(x_0, y_0)$ $\qquad f(x,y) = 1-y$

$= 0 + 0.1 f(0,0)$ $\qquad f(0,0) = 1-0 = 1$

$= 0.1(1)$

$y_1 = 0.1$

$y_2 = y_1 + h f(x_1, y_1)$

$= 0.1 + 0.1 f(0.1, 0.1)$

$= 0.1 + 0.1 [1 - 0.1]$

$= 0.1 + 0.09$

$y_2 = 0.19$

$y_2 = y(x_2) = y(0.2) = 0.19$

Ex. $y' = -y$, $y(\underset{x_0}{0}) = \underset{y_0}{1.5}$ at the point $x = 0.04$, $h = 0.01$

$\Rightarrow \dfrac{dy}{dx} = -y$

$\Rightarrow f(x,y) = -y$, $x_0 = 0$, $y_0 = 1.5$

$\underset{y_0}{x_0 = 0}$, $\underset{y_1}{x_1 = 0.01}$, $\underset{y_2}{x_2 = 0.02}$, $\underset{y_3}{x_3 = 0.03}$, $\underset{y_4}{x_4 = 0.04}$

$y_1 = y_0 + h f(x_0, y_0)$

$= 1.5 + 0.01 f(0, 1.5)$

$= 1.5 + 0.01(-1.5)$

$= 1.485$

$y_2 = y_1 + h f(x_1, y_1)$

$= 1.485 + 0.01 f(0.01, 1.485)$

$= 1.485 + 0.01 [-1.485]$

$= 1.4701$

$$y_3 = y_2 + h f(x_2, y_2)$$
$$= 1.4701 + 0.01 f(0.02, 1.4701)$$
$$= 1.4701 + 0.01 (-1.4701)$$
$$= 1.4554$$

$$y_4 = y_3 + h f(x_3, y_3)$$
$$= 1.4554 + 0.01 f(0.03, 1.4554)$$
$$= 1.4554 + 0.01 (-1.4554)$$
$$y_4 = 1.4408$$

i.e. $y(0.04) = 1.4408$.

Ex. $y'(x+y) = y - x$, $y(0) = 2$, find the value of $y(0.02)$, $y(0.04)$, $y(0.06)$.

$\Rightarrow$ $y'(x+y) = y - x$

$$y' = \frac{y-x}{x+y}$$

$$\frac{dy}{dx} = \frac{y-x}{x+y}$$

$$f(x,y) = \frac{y-x}{x+y}, \quad x_0 = 0, \; y_0 = 2$$

Take $h = 0.02$

$x_0 = 0$, $x_1 = 0.02$, $x_2 = 0.04$, $x_3 = 0.06$

find, y_1, y_2, y_3.

$$y_1 = y_0 + h f(x_0, y_0)$$
$$= 2 + 0.02 f(0, 2)$$
$$= 2 + 0.02 \left[\frac{2-0}{0+2}\right]$$
$$= 2 + 0.02$$
$$= 2.02$$

* Picard's Method of successive Approximations –

Consider the differential eqⁿ

$$\frac{dy}{dx} = f(x,y) \quad \text{—①}$$

with initial condition

$$y(x_0) = y_0$$

These types of equations can be solved by the method of successive approximations.

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx$$

$$y^{(3)} = y_0 + \int_{x_0}^{x} f(x, y^{(2)})\, dx$$

$$\vdots$$

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\, dx$$

This method gives a sequence of approximations $y^{(1)}, y^{(2)}, y^{(3)}, \ldots y^{(n)}$ gives the approximate solutions of the given diff. eqⁿ.

Remark – This method gives a sequence of approximations $y^{(1)}, y^{(2)}, \ldots, y^{(n)}$ it can be proved that if the function $f(x,y)$ is bounded in some region about the point (x_0, y_0) and if $f(x,y)$ satisfies the Lipschitz condition

$$|f(x,y) - f(x,\bar{y})| \le k\, f(x,y)$$

where k is const then the seqⁿ $y^{(1)}, y^{(2)}, \ldots y^{(n)}$ converges to the solution of eqⁿ ①

Ex. Use Picard's method to obtain a series soln of the diff. eqn $\frac{dy}{dx} = 1 + xy$ with initial condn $y(0)=1$. obtain y for $x = 0.1$ correct to four decimal places.

$\Rightarrow$ $\frac{dy}{dx} = 1 + xy$, $y(0) = 1$

Take $f(x,y) = 1 + xy$, $x_0 = 0$ & $y_0 = 1$

$f(x, y_0) = f(x, 1) = 1 + x$

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$= 1 + \int_0^x (1+x)\, dx$$

$$= 1 + \left[x + \frac{x^2}{2}\right]_{x=0}^{x}$$

$$y^{(1)} = 1 + x + \frac{x^2}{2} - 0 = 1 + x + \frac{x^2}{2}$$

$$f(x, y^{(1)}) = 1 + x\left(1 + x + \frac{x^2}{2}\right) = 1 + x + x^2 + \frac{x^3}{2}$$

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx$$

$$= 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{2}\right) dx$$

$$= 1 + \left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\right]_{x=0}^{x}$$

$$y^{(2)} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

$$f(x, y^{(2)}) = 1 + x\left[1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\right]$$

$$= 1 + x + x^2 + \frac{x^3}{2} + \frac{x^4}{3} + \frac{x^5}{8}$$

$$y^{(3)} = y_0 + \int_{x_0}^{x} f(x, y^{(2)})\, dx$$

$$= 1 + \int_{x_0}^{x} \left(1 + x + x^2 + \frac{x^3}{2} + \frac{x^4}{3} + \frac{x^5}{8}\right) dx$$

$$= 1 + \left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48} \right]_{x=0}^{x}$$

$$y^{(3)} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}$$

This is required soln of given diff eqn.

Now to find the value of $y(0.1)$ correct upto four decimal places consider the term upto x^4

$$y = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

$$y(0.1) = 1 + 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{4}$$

$$= 1.1053.$$

Ex. solve $y' = 2x + 3y$, $y(0) = 1$ & find $y(0.25)$ by using Picard's method (do three steps)

$\Rightarrow$ $y' = 2x + 3y$, $y(0) = 1$

$$f(x,y) = 2x + 3y, \quad x_0 = 0, \; y_0 = 1$$

$$f(x, y_0) = f(x, 1) = 2x + 3$$

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$= 1 + \int_0^x (2x + 3)\, dx$$

$$= 1 + \left[\cancel{2}\frac{x^2}{\cancel{2}} + 3x \right]_{x=0}^{x}$$

$$y^{(1)} = 1 + x^2 + 3x$$

$$f(x, y^{(1)}) = 2x + 3(1 + x^2 + 3x)$$

$$= 2x + 3 + 3x^2 + 9x$$

$$= 11x + 3x^2 + 3 = 3 + 11x + 3x^2$$

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx$$

$$= 1 + \int_0^x (3 + 11x + 3x^2)\,dx$$

$$= 1 + \left[3x + 11\frac{x^2}{2} + x^3\right]_{x=0}^{x}$$

$$y^{(2)} = 1 + 3x + \frac{11}{2}x^2 + x^3$$

$$f(x, y^{(2)}) = 2x + 3\left[1 + 3x + \frac{11}{2}x^2 + x^3\right]$$

$$= 2x + 3 + 9x + \frac{33}{2}x^2 + 3x^3$$

$$= 3x^3 + \frac{33}{2}x^2 + 11x + 3$$

$$y^{(3)} = y_0 + \int_{x_0}^{x} f(x, y^{(2)})\,dx$$

$$= 1 + \int_0^x \left[3x^3 + \frac{33}{2}x^2 + 11x + 3\right]dx$$

$$= 1 + \left[3\frac{x^4}{4} + \frac{11}{2}x^3 + 11\frac{x^2}{2} + 3x\right]_{x=0}^{x}$$

$$y^{(3)} = 1 + \frac{3}{4}x^4 + \frac{11}{2}x^3 + \frac{11}{2}x^2 + 3x$$

$\therefore$ This soln of given eqn eqn

$$y(0.25) = 1 + \frac{3}{4}(0.0039) + \frac{11}{2}(0.0156) + 0.75$$

$$= 1 + 0.0029 + 0.0858 + 0.75$$

$$= 1.8387$$

Ex. Using Picard's method find $y(0.1)$ for solving the diff eqn $\frac{dy}{dx} = \frac{y-x}{y+x}$ with initial condn $y(0) = 1$

$y(0) = 1$
$\uparrow x_0 \quad \uparrow y_0$

$$\Rightarrow \quad f(x,y) = \frac{y-x}{y+x}, \quad x_0 = 0, \quad y_0 = 1$$

$$f(x, y_0) = f(x, 1) = \frac{1-x}{1+x}$$

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$= 1 + \int_0^x \frac{1-x}{1+x}\, dx$$

$$= 1 + \int_0^x \frac{1-x-1+1}{1+x}\, dx$$

$$= 1 + \int_0^x \frac{2-(x+1)}{1+x}\, dx$$

$$= 1 + \int_0^x \left[\frac{2}{1+x} - 1\right] dx$$

$$= 1 + \left[2\ln(1+x) - x\right]_{x=0}^{x}$$

$\ln(1) = 0$

$$y^{(1)} = 1 + 2\ln(1+x) - x - 0$$

$$y(0.1) = 1 + 2\ln(1+0.1) - 0.1$$

$$= 0.9 + 0.1906$$

$$= 1.0906$$

Ex. Using Picard's method find $y(0.25)$, $y(0.50)$ & $y(1.0)$ for solving the following diff eq^n $\frac{dy}{dx} = \frac{x^2}{y^2+1}$ with initial condition $y(0) = 0$.

$\Rightarrow$ $f(x, y) = \frac{x^2}{y^2+1}$, $x_0 = 0$, $y_0 = 0$

$$f(x, y_0) = \frac{x^2}{0+1} = x^2$$

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$= 0 + \int_0^x x^2\, dx$$

$$= \left[\frac{x^3}{3}\right]_{x=0}^{x}$$

$$y^{(1)} = \frac{x^3}{3}$$

$$y(0.25) = 0.005$$

$$y(0.50) = 0.0417$$

$$y(1.0) = 0.3333$$

Ex. $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$, $y(1) = 1$, find $y(1.1)$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x^2} - \frac{y}{x} \quad , \quad y(1) = 1$$

$$\therefore \quad f(x,y) = \frac{1}{x^2} - \frac{y}{x} \quad , \quad x_0 = 1 \, , \; y_0 = 1$$

$$f(x,y_0) = f(x,1) = \frac{1}{x^2} - \frac{1}{x}$$

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x,y_0)\,dx$$

$$= 1 + \int_{1}^{x} \left[\frac{1}{x^2} - \frac{1}{x}\right] dx$$

$$= 1 + \left[-\frac{1}{x} - \ln x\right]_{x=1}^{x}$$

$$= 1 + \left(-\frac{1}{x} - \ln x\right) - (-1 - \ln 1)$$

$$= 1 - \frac{1}{x} - \ln x + 1 + 0$$

$$y^{(1)} = 2 - \frac{1}{x} - \ln x$$

$$y(1.1) = 2 - \frac{1}{1.1} - \ln(1.1)$$

$$= 2 - 0.9090 - 0.0953$$

$$= 0.9957.$$

Chapter 4

Numerical solution of first order ordinary differential equations.

* Taylor's series Method –

consider the differential eqⁿ

$$\frac{dy}{dx} = f(x,y) \qquad \text{—— (1)}$$

$$\text{or} \quad y' = f(x,y)$$

with initial condition

$$y(x_0) = y_0 \qquad \text{—— (2)}$$

If $y(x)$ is exact solution of eqⁿ (1) then the Taylor's series for $y(x)$ around $x = x_0$ is given by

$$y(x) = y(x_0) + (x-x_0)\left(\frac{dy}{dx}\right)_{x=x_0} + \frac{1}{2!}(x-x_0)^2\left(\frac{d^2y}{dx^2}\right)_{x=x_0} + \frac{1}{3!}(x-x_0)^3\left(\frac{dy}{dx}\right)_{x=x_0} + \cdots$$

$$\therefore\ y(x) = y_0 + (x-x_0)y'_0 + \frac{1}{2!}(x-x_0)^2 y''_0 + \frac{1}{3!}(x-x_0)^3 y'''_0 + \cdots \qquad \text{—— (3)}$$

If the values $y'_0, y''_0, y'''_0, \ldots$ are known then eqⁿ (3) gives a power series for y.

Remark – Taylor's series method can not be applied to all problems as we need the higher derivatives & computation of higher order derivatives is tedious task.

Ex. Solve the following initial value problem using Taylor's series method. Also find the value of y for given x.

1) $y' = x^2 + y^2$, $y(0) = 1$, find $y(0.1)$

$\Rightarrow$ Here $x_0 = 0$, $y_0 = 1$, $x = 0.1$

$$y'(x) = x^2 + y^2 \Rightarrow y_0' = y'(0) = x_0^2 + [y(0)]^2$$
$$= 0^2 + 1^2 = 1$$
$$\Rightarrow y_0' = 1$$

$$y'' = 2x + 2yy' \Rightarrow y_0'' = y_0''(0) = 2(0) + 2y_0 y_0'$$
$$= 2(1)(1) = 2$$

$$y''' = 2 + 2y'^2 + 2yy'' \Rightarrow y_0''' = 2 + 2y_0'^2 + 2y_0 y_0''$$
$$= 2 + 2(1) + 2(1)(2)$$
$$= 8$$

$$y'''' = 4y'y'' + 2y'y'' + 2yy'''$$
$$\Rightarrow y_0'''' = 4(1)(2) + 2(1)(2) + 2(1)(8)$$
$$= 12 + 16 = 28$$

Taylor's series for $y(x)$ is given by

$$y(x) = y_0 + (x - x_0) y_0' + \frac{1}{2!}(x - x_0)^2 y_0'' + \frac{1}{3!}(x - x_0)^3 y_0''' + \cdots$$

here $x_0 = 0$

$$y(x) = 1 + x(1) + \frac{1}{2} x^2 (2) + \frac{1}{6} x^3 (28) + \cdots$$

put $x = 0.1$

$$y(0.1) = 1 + 0.1 + 0.01 + 0.0047$$
$$= 1.1147$$

2) Solve the differential eqn $y' = x - y^2$ with initial condition $y(0) = 1$ by Taylor's series method and find $y(0.1)$ correct to four decimal places.

$\Rightarrow$ $y' = x - y^2$

$y(0) = 1$

$\therefore x_0 = 0$ & $y_0 = 1$

$$y' = x - y^2 \Rightarrow y_0' = y'(x_0) = x_0 - [y(x_0)]^2$$
$$\Rightarrow y_0' = y'(0) = 0 - [y(0)]^2 = -(1)^2 = -1$$

$$y'' = 1 - 2yy' \Rightarrow y_0'' = 1 - 2y_0y_0'$$

$$= 1 - 2(1)(-1) = 3$$

$uv = uv' + u'v$

$$y''' = 0 - 2yy'' - 2y'y' = -2yy'' - 2y'^2$$

$$\Rightarrow y_0''' = -2y_0y_0'' - 2y_0'^2$$

$$= -2(1)(3) - 2(-1)^2$$

$$= -6 - 2 = -8$$

$y'^2 = 2y'y''$

$$y^{iv} = -2y'y'' - 2yy''' - 4y'y''$$

$$= -6y'y'' - 2yy''' \Rightarrow y_0^{iv} = -6y_0'y_0'' - 2y_0y_0'''$$

$$= -6(-1)(3) - 2(1)(-8)$$

$$= 18 + 16 = 34$$

$$y^{v} = -6y'y''' - 6y''y'' - 2y'y''' - 2yy^{iv}$$

$$= -8y'y''' - 6(y'')^2 - 2yy^{iv}$$

$$\Rightarrow y_0^{v} = -8y_0'y_0''' - 6(y_0'')^2 - 2y_0y_0^{iv}$$

$$= -8(-1)(-8) - 6(3)^2 - 2(1)(34)$$

$$= -64 - 54 - 68$$

$$= -186$$

Taylor's series of $y(x)$ at $x = x_0$ is

$$y(x) = y_0 + (x - x_0)y_0' + \frac{1}{2!}(x - x_0)^2 y_0'' + \frac{1}{3!}(x - x_0)^3 y_0'''$$

$$+ \frac{1}{4!}(x - x_0)^4 y_0^{iv} + \frac{1}{5!}(x - x_0)^5 y_0^{v} + \cdots$$

here $x_0 = 0$

$$y(x) = 1 + x(-1) + \frac{1}{2}x^2(3) + \frac{1}{6}x^3(-8)$$

$$+ \frac{1}{24}x^4(34) + \frac{1}{120}x^5(-186) + \cdots$$

This is required soln of the given diff. eqn

To find the value of $y(0.1)$ correct upto four decimal places we consider term upto x^4 of above series

$$y(0.1) = 1 - 0.1 + \frac{3}{2}(0.1)^2 - \frac{8}{6}(0.1)^3 + \frac{34}{24}(0.1)^4$$

$$= 1 - 0.1 + 0.015 - 0.0013 + 0.0001$$

$$= 0.9138$$

Remark - If we want to find the range of x for which the above series, truncated after the term containing x^4, can be used to compute the value of y correct upto four decimal palces,

we write

$$\frac{186}{120} x^5 \leq 0.00005$$

$$x^5 \leq 3.2258 \times 10^{-5}$$

$$\Rightarrow x \leq 0.126$$

Ex. Solve the following diff eq^n $\frac{dy}{dx} = x+y$ with initial condition $y(1) = 0$, by Taylor's series method & compute $y(1.1)$ correct upto four decimal places.

$\Rightarrow$ $y' = x+y$, $y(\underset{\uparrow x_0}{1}) = \underset{\uparrow y_0}{0}$

$x_0 = 1$ & $y_0 = 0$

$y' = x+y \Rightarrow y'_0 = y'(x_0) = y'(1) = 1 + y(1) = 1+0 = 1$

$y'' = 1+y' \Rightarrow y''_0 = y''(x_0) = y''(1) = 1 + y'_0$

$= 1+1 = 2$

$y''' = 0 + y'' \Rightarrow y'''_0 = y''_0 = 2$

$y^{iv} = y''' \Rightarrow y^{iv}_0 = y'''_0 = 2$

Taylor's series for $y(x)$ at $x = x_0$

$$y(x) = y_0 + (x-x_0)\,y_0' + \frac{1}{2!}(x-x_0)^2\,y_0'' + \frac{1}{3!}(x-x_0)^3\,y_0''' + \frac{1}{4!}(x-x_0)^4\,y_0^{iv} + \cdots$$

here $x_0 = 1$

$$y(x) = 0 + (x-1)(1) + \frac{1}{2}(x-1)^2(2) + \frac{1}{6}(x-1)^3(2) + \frac{1}{24}(x-1)^4(2) + \cdots$$

$$= (x-1) + (x-1)^2 + \frac{1}{3}(x-1)^3 + \frac{1}{12}(x-1)^4 + \cdots$$

This is series soln of given diff eqn

$$y(1.1) = 0.1 + 0.01 + 0.0003 + 0.000008$$

$$= 0.1103$$

Ex. Solve the following differential eqn $\frac{dy}{dx} - 2y = 3e^x$, with initial condition $y(0) = 0$ by Taylor's series method & compute $y(0.1)$

$$\Rightarrow \quad \frac{dy}{dx} = 3e^x + 2y$$

$$y' = 3e^x + 2y, \quad y(0) = 0$$

here $x_0 = 0$ & $y_0 = 0$

or $(3e^0 + 2y_0)$

$$y' = 3e^x + 2y \Rightarrow y_0' = y'(x_0) = y'(0) = 3e^0 + 2y(0)$$

$$= 3 + 2(0) = 3$$

$$y'' = 3e^x + 2y' \Rightarrow y_0'' = y''(0) = 3e^0 + 2y_0'$$

$$= 3 + 2(3)$$

$$= 9$$

$$y''' = 3e^x + 2y'' \Rightarrow y_0''' = y'''(0) = 3e^0 + 2y_0''$$
$$= 3 + 2(9)$$
$$= 21$$

$$y^{iv} = 3e^x + 2y''' \Rightarrow y_0^{iv} = 3e^0 + 2y_0'''$$
$$= 3 + 2(21)$$
$$= 45$$

$$y(x) = y_0 + (x - x_0)y_0' + \frac{1}{2!}(x-x_0)^2 y_0'' + \frac{1}{3!}(x-x_0)^3 y_0''' + \frac{1}{4!}(x-x_0)^4 y_0^{iv} + \cdots$$

here $x_0 = 0$

$$y(x) = 0 + x(3) + \frac{1}{2}x^2(9) + \frac{1}{6}x^3(21) + \frac{1}{24}x^4(45) + \cdots$$

$$= 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \frac{45}{24}x^4 + \cdots$$

Find $y(0.1)$

$$y(0.1) = 0.3 + 0.045 + 0.0035 + 0.0001$$
$$= 0.3486$$

Ex. Evaluate $y(0.1)$ using Taylor's series method, given $y'' - x(y')^2 + y^2 = 0, y(0) = 1$ & $y'(0) = 0$.

$$\Rightarrow \quad y'' - xy'^2 + y^2 = 0$$
$$y'' = xy'^2 - y^2, \quad y(0) = 1 \text{ \& } y'(0) = 0$$

$x_0 = 0$ & $y_0 = y(x_0) = y(0) = 1$

& $y_0' = y'(x_0) = y'(0) = 0$

$$y'' = xy'^2 - y^2 \Rightarrow y_0'' = y''(x_0) = y''(0) = 0 - [y(0)]^2$$
$$= -1$$

$$y''' = x2y'y'' + y'^2(1) - 2yy'$$

$$= 2xy'y'' + (y')^2 - 2yy'$$

$$\Rightarrow y'''_0 = y'''(0) = 0 + 0 + 0 = 0$$

$$y^{iv} = 2y'y'' + 2xy''y'' + 2xy'y''' + 2y'y'' - 2y'y' - 2yy''$$

$$\Rightarrow y^{iv}_0 = y^{iv}(0) = 0 + 0 + 0 + 0 - 2(0) - 2(1)(-1)$$

$$= 2$$

Taylor's series

$$y(x) = y_0 + (x - x_0)y'_0 + \frac{1}{2!}(x - x_0)^2 y''_0 + \frac{1}{3!}(x - x_0)^3 y'''_0$$

$$+ \frac{1}{4!}(x - x_0)^4 y^{iv}_0 + \cdots$$

$$y(x) = 1 + \frac{1}{2}x^2(-1) + \frac{1}{24}x^4(2) + \cdots$$

$$= 1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 + \cdots$$

$$y(0.1) = 1 - 0.005 + 0.000008$$

$$= 0.9950$$

Reference: Numerical methods and its applications, text book for S.Y.B.Sc., by Golden series, Nirali Prakashan