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Chapter 4: Linear Transformation

Topic- Definition and properties of linear transformation

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Chapter 4
Linear Transformation

* Def^n - Linear Transformation (or mapping or function)

Let V & W be two real vector spaces. A function $T: V \to W$ is said to be linear transformation if it satisfy following axioms for all vector $\bar{u}, \bar{v}$ in V and scalar k

a) $T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v})$

b) $T(k\bar{u}) = k\,T(\bar{u})$

Thm A function $T: V \to W$ is linear transformation (L.T) if and only if

$$T(k_1\bar{u}_1 + k_2\bar{u}_2) = k_1 T(\bar{u}_1) + k_2 T(\bar{u}_2)$$

for any vectors $\bar{u}_1$ & $\bar{u}_2$ in V and any scalars k_1 & k_2.

Proof - Suppose T is L.T.

consider $T(k_1\bar{u}_1 + k_2\bar{u}_2)$

$$= T(k_1\bar{u}_1) + T(k_2\bar{u}_2)$$

$$= k_1 T(\bar{u}_1) + k_2 T(\bar{u}_2)$$

conversely, consider

$$T(k_1\bar{u}_1 + k_2\bar{u}_2) = k_1 T(\bar{u}_1) + k_2 T(\bar{u}_2)$$

TST T is linear transformation (L.T.)

a) Take $k_1 = k_2 = 1$ & $\bar{u}_1 = \bar{u}$ & $\bar{u}_2 = \bar{v}$

$\therefore T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v})$

b) $k_1 = k$, $k_2 = 0$, $\bar{u}_1 = \bar{u}$

$\therefore T(k\bar{u}) = k\,T(\bar{u})$

$\therefore$ T is L.T.

Remark In general

$$T(k_1\bar{u}_1 + k_2\bar{u}_2 + \cdots + k_n\bar{u}_n) = k_1 T(\bar{u}_1) + k_2 T(\bar{u}_2) + \cdots + k_n T(\bar{u}_n)$$

Ex. Determine whether the following transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear, if not justify?

a) $T(x,y) = (x+2y, 3x-y)$

$\Rightarrow$ consider $\bar{u} = (x,y)$, $\bar{v} = (x', y')$

$\bar{u}+\bar{v} = (x+x', y+y')$

a) $T(\bar{u}+\bar{v}) = T(x+x', y+y')$

$= (x+x'+2(y+y'), 3(x+x')-(y+y'))$

$= (x+x'+2y+2y', 3x+3x'-y-y')$

$T(\bar{u})+T(\bar{v}) = T(x,y) + T(x',y')$

$= (x+2y, 3x-y) + (x'+2y', 3x'-y')$

$= (x+2y+x'+2y', 3x-y+3x'-y')$

$\therefore T(\bar{u}+\bar{v}) = T(\bar{u})+T(\bar{v})$

b) $k\cdot\bar{u} = (kx, ky)$

consider $T(k\bar{u}) = T(kx, ky)$

$= (kx+2ky, 3kx-ky)$

$= k(x+2y, 3x-y)$

$= k\,T(x,y)$

$= k\,T(\bar{u})$

$\therefore$ T is L.T.

2) $T(x,y) = (x, y^2)$

$\Rightarrow$ consider $\bar{u} = (x,y)$, $\bar{v} = (x',y')$, $\bar{u}+\bar{v} = (x+x', y+y')$

a) $T(\bar{u}+\bar{v}) = T(x+x', y+y')$

$= (x+x', (y+y')^2)$

$= (x+x', y^2+2yy'+y'^2) \longrightarrow 1$

$T(\bar{u}) + T(\bar{v}) = T(x,y) + T(x',y')$

$= (x, y^2) + (x', y'^2)$

$= (x+x', y^2+y'^2) \longrightarrow 2$

from ① & ②

$$T(\bar{u}+\bar{v}) \neq T(\bar{u}) + T(\bar{v})$$

$\therefore$ T is not L.T.

3] $T(x,y) = (x,0)$

$\Rightarrow$ $\bar{u} = (x,y)$, $\bar{v} = (x',y')$, $\bar{u}+\bar{v} = (x+x', y+y')$

ⓐ $T(\bar{u}+\bar{v}) = T(x+x', y+y')$

$= (x+x', 0)$

$T(\bar{u}) + T(\bar{v}) = T(x,y) + T(x',y')$

$= (x,0) + (x',0)$

$= (x+x', 0)$

$\therefore$ $T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v})$

ⓑ $k \cdot \bar{u} = (kx, ky)$

$T(k\bar{u}) = T(kx, ky)$

$= (kx, 0)$

$= k(x, 0)$

$= kT(\bar{u})$

$\therefore$ T is L.T.

Ex. Determine whether the following transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ is linear, if not justify?

1) $T(x,y,z) = (x, yz)$

$\Rightarrow$ $\bar{u} = (x,y,z)$, $\bar{v} = (x',y',z')$, $\bar{u}+\bar{v} = (x+x', y+y', z+z')$

a) consider

$T(\bar{u}+\bar{v}) = T(x+x', y+y', z+z')$

$= (x+x', (y+y')(z+z'))$

$= (x+x', yz+yz'+y'z+y'z')$ —①

$T(\bar{u}) + T(\bar{v}) = T(x,y,z) + T(x',y',z')$

$= (x,yz) + (x',y'z')$

$= (x+x', yz+y'z')$ —②

From ① & ② $T(\bar{u}+\bar{v}) \neq T(\bar{u})+T(\bar{v})$

$\therefore$ T is not a L.T.

2) $T(x,y,z) = (1,1)$

$\Rightarrow$ $\bar{u} = (x,y,z)$, $\bar{v} = (x',y',z')$, $\bar{u}+\bar{v} = (x+x', y+y', z+z')$

Consider

$T(\bar{u}+\bar{v}) = T(x+x', y+y', z+z')$

$= (1,1)$ — ①

$T(\bar{u})+T(\bar{v}) = T(x,y,z)+T(x',y',z')$

$= (1,1)+(1,1)$

$= (2,2)$ — ②

$\therefore$ from ① & ②

T is not a L.T.

Ex. $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

1) $T(x,y,z) = (x-y, y-z, z-x)$

$\bar{u} = (x,y,z)$, $\bar{v} = (x',y',z')$, $\bar{u}+\bar{v} = (x+x', y+y', z+z')$

Consider

$T(\bar{u}+\bar{v}) = (x+x', y+y', z+z')$

$= (x+x'-(y+y'), y+y'-(z+z'), z+z'-(x+x'))$

$= (x+x'-y-y', y+y'-z-z', z+z'-x-x')$

$T(\bar{u})+T(\bar{v}) = T(x,y,z)+T(x',y',z')$

$= (x-y, y-z, z-x)+(x'-y', y'-z', z'-x')$

$= (x-y+x'-y', y-z+y'-z', z-x+z'-x')$

$\therefore T(\bar{u}+\bar{v}) = T(\bar{u})+T(\bar{v})$

ⓑ $k\cdot\bar{u} = (kx, ky, kz)$

$T(k\cdot\bar{u}) = T(kx, ky, kz)$

$= (kx-ky, ky-kz, kz-kx)$

$= k(x-y, y-z, z-x) = kT(x,y,z)$

$= kT(\bar{u})$

$\therefore T$ is L.T.

2) $T(x,y,z) = (x+1,\ y+1,\ z+1)$

$\bar{u} = (x,y,z)\ ,\ \bar{v} = (x',y',z')$

$\bar{u}+\bar{v} = (x+x',\ y+y',\ z+z')$

$$T(\bar{u}+\bar{v}) = T(x+x',\ y+y',\ z+z')$$
$$= (x+x'+1,\ y+y'+1,\ z+z'+1) \quad (1)$$

$$T(\bar{u})+T(\bar{v}) = T(x,y,z) + T(x',y',z')$$
$$= (x+1,\ y+1,\ z+1) + (x'+1,\ y'+1,\ z'+1)$$
$$= (x+x'+2,\ y+y'+2,\ z+z'+2) \quad (2)$$

$$T(\bar{u}+\bar{v}) \neq T(\bar{u})+T(\bar{v})$$

$\therefore T$ is not L.T.

3) $T(x,y,z) = (e^{x},\ e^{y},\ 0)$

$\bar{u} = (x,y,z)\ ,\ \bar{v} = (x',y',z')\ ,\ \bar{u}+\bar{v} = (x+x',\ y+y',\ z+z')$

a) $T(\bar{u}+\bar{v}) = T(x+x',\ y+y',\ z+z')$
$$= (e^{x+x'},\ e^{y+y'},\ 0)$$

$$T(\bar{u})+T(\bar{v}) = T(x,y,z) + T(x',y',z')$$
$$= (e^{x},\ e^{y},\ 0) + (e^{x'},\ e^{y'},\ 0)$$
$$= (e^{x}+e^{x'},\ e^{y}+e^{y'},\ 0)$$

Since $e^{x+x'} \neq e^{x}+e^{x'}$

$\therefore T(\bar{u}+\bar{v}) = T(\bar{u})+T(\bar{v})$

$\therefore T$ is not a L.T.

* Identity transformation -

The transformation $I: V \to V$ defined as $I(\bar{u}) = \bar{u}$ for every $\bar{u} \in V$ is a linear transformation from V to V. This transformation is called as identity transformation on V.

* Zero (Null) transformation -

Let V & W be any two vector spaces. The mapping $T: V \to W$ defined by $T(\bar{u}) = \bar{0}$, $\forall\ \bar{u} \in V$ is a linear transformation called the zero transformation.

* Equal transformation -

Two transformations T_1 & T_2 from V to W are equal if and only if $T_1(\bar{u}) = T_2(\bar{u})$ for every $\bar{u} \in V$.

Thm - If $\{\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_n\}$ is a basis for V and $\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_n$ are arbitrary n vectors in W, not necessarily distinct, then there exist a unique linear transformation $T: V \to W$ such that

$$T(\bar{v}_i) = \bar{w}_i \quad , \quad i = 1, 2, \ldots n.$$

Proof - Since $\{\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_n\}$ is basis for V

$\therefore$ for every $\bar{u} \in V$, $\bar{u}$ can be written as

$$\bar{u} = k_1\bar{v}_1 + k_2\bar{v}_2 + \cdots + k_n\bar{v}_n$$

Define, $T: V \to W$ as

$$T(\bar{u}) = k_1\bar{w}_1 + k_2\bar{w}_2 + \cdots + k_n\bar{w}_n$$

$T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v})$
$T(k\bar{u}) = kT(\bar{u})$

TST T is L.T.

(a) Let $\bar{u}, \bar{v} \in V$

$\therefore \bar{u} = k_1\bar{v}_1 + k_2\bar{v}_2 + \cdots + k_n\bar{v}_n$

$\bar{v} = c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_n\bar{v}_n$

$\therefore \bar{u}+\bar{v} = (k_1+c_1)\bar{v}_1 + (k_2+c_2)\bar{v}_2 + \cdots + (k_n+c_n)\bar{v}_n$

$T(\bar{u}+\bar{v}) = (k_1+c_1)\bar{\omega}_1 + (k_2+c_2)\bar{\omega}_2 + \cdots + (k_n+c_n)\bar{\omega}_n \quad —(1)$

Now $T(\bar{u}) = k_1\bar{\omega}_1 + k_2\bar{\omega}_2 + \cdots + k_n\bar{\omega}_n$

$T(\bar{v}) = c_1\bar{\omega}_1 + c_2\bar{\omega}_2 + \cdots + c_n\bar{\omega}_n$

$\therefore T(\bar{u}) + T(\bar{v}) = (k_1+c_1)\bar{\omega}_1 + \cdots + (k_n+c_n)\bar{\omega}_n \quad —(2)$

from (1) & (2)

$\therefore T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v})$

(b) $\alpha\bar{u} = \alpha k_1\bar{v}_1 + \alpha k_2\bar{v}_2 + \cdots + \alpha k_n\bar{v}_n$

$T(\alpha\bar{u}) = \alpha k_1\bar{\omega}_1 + \alpha k_2\bar{\omega}_2 + \cdots + \alpha k_n\bar{\omega}_n$

$= \alpha[k_1\bar{\omega}_1 + k_2\bar{\omega}_2 + \cdots + k_n\bar{\omega}_n]$

$T(\alpha\bar{u}) = \alpha T(\bar{u})$

$\therefore$ T is Linear transformation.

Uniqueness -

Suppose there are two linear transformations T & T' such that

$T(\bar{v}_i) = \bar{\omega}_i \quad , \quad i = 1, 2 \ldots n$

$T'(\bar{v}_i) = \bar{\omega}_i \quad$ —||—

Then for $\bar{u} \in V$

$T(\bar{u}) = k_1\bar{\omega}_1 + k_2\bar{\omega}_2 + \cdots + k_n\bar{\omega}_n$

$= k_1 T'(\bar{v}_1) + k_2 T'(\bar{v}_2) + \cdots + k_n T'(\bar{v}_n)$

$= T'(k_1\bar{v}_1) + T'(k_2\bar{v}_2) + \cdots + T'(k_n\bar{v}_n)$

$= T'(k_1\bar{v}_1 + k_2\bar{v}_2 + \cdots + k_n\bar{v}_n)$

$= T'(\bar{u})$

$T(\bar{u}) = T'(\bar{u})$

$\therefore T = T'$

Thm If $T: V \to W$ is a L.T. then

a) $T(\bar{0}) = \bar{0}$

b) $T(-\bar{u}) = -T(\bar{u})$, $\forall \bar{u} \in V$

c) $T(\bar{u} - \bar{v}) = T(\bar{u}) - T(\bar{v})$, $\forall \bar{u}, \bar{v} \in V$

Proof -

a) $T(\bar{0}) = T(0 \cdot \bar{u})$
$= 0\, T(\bar{u})$
$= \bar{0}$

b) $T(-\bar{u}) = T(-1 \cdot \bar{u})$
$= -1\, T(\bar{u})$
$= -T(\bar{u})$

c) $T(\bar{u} - \bar{v}) = T(\bar{u} + (-1)\bar{v})$
$= T(\bar{u}) + T(-1 \cdot \bar{v})$
$= T(\bar{u}) - T(\bar{v})$

Ex. Let $T: V \to \mathbb{R}^3$ be a L.T. & $\bar{v}_1, \bar{v}_2, \bar{v}_3$ are vectors in V s.t. $T(\bar{v}_1) = (2,-1,2)$, $T(\bar{v}_2) = (3,0,2)$, $T(\bar{v}_3) = (-2,1,3)$ Find $T(5\bar{v}_1 - 2\bar{v}_2 + \bar{v}_3)$

$\Rightarrow$ $T(5\bar{v}_1 - 2\bar{v}_2 + \bar{v}_3) = 5T(\bar{v}_1) - 2T(\bar{v}_2) + T(\bar{v}_3)$
$= 5(2,-1,2) - 2(3,0,2) + (-2,1,3)$
$= (10,-5,10) - (6,0,4) + (-2,1,3)$
$= (10-6-2, -5-0+1, 10-4+3)$
$= (2,-4,9)$

Ex. Consider the basis $S = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ for $\mathbb{R}^3$ where $\bar{v}_1 = (1,1,1)$, $\bar{v}_2 = (1,1,0)$, $\bar{v}_3 = (1,0,0)$

and let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a L.T. such that
$T(\bar{v}_1) = (1,0)$, $T(\bar{v}_2) = (2,-1)$, $T(\bar{v}_3) = (4,3)$
Find a formula $T(x_1, x_2, x_3)$ and use it to compute $T(2,-3,5)$.

$\Rightarrow$ Consider

$$(x_1, x_2, x_3) = k_1\bar{v}_1 + k_2\bar{v}_2 + k_3\bar{v}_3$$
$$= k_1(1,1,1) + k_2(1,1,0) + k_3(1,0,0)$$
$$= (k_1+k_2+k_3,\ k_1+k_2,\ k_1)$$

$\therefore$ $k_1 + k_2 + k_3 = x_1$ — ①

$k_1 + k_2 = x_2$ — ②

$k_1 = x_3$

From ②

$x_3 + k_2 = x_2 \Rightarrow k_2 = x_2 - x_3$

From ①

$x_3 + x_2 - x_3 + k_3 = x_1$

$k_3 = x_1 - x_2$

$\therefore (x_1, x_2, x_3) = x_3\bar{v}_1 + (x_2 - x_3)\bar{v}_2 + (x_1 - x_2)\bar{v}_3$

$\therefore T(x_1, x_2, x_3) = x_3 T(\bar{v}_1) + (x_2 - x_3) T(\bar{v}_2) + (x_1 - x_2) T(\bar{v}_3)$

$= x_3(1,0) + (x_2 - x_3)(2,-1) + (x_1 - x_2)(4,3)$

$= (x_3 + 2x_2 - 2x_3 + 4x_1 - 4x_2,\ 0 - x_2 + x_3 + 3x_1 - 3x_2)$

$T(x_1, x_2, x_3) = (4x_1 - 2x_2 - x_3,\ 3x_1 - 4x_2 + x_3)$

$\therefore T(2,-3,5) = (8 + 6 - 5,\ 6 + 12 + 5)$

$= (9, 23)$

Ex. Consider the basis $S = \{\bar{v}_1, \bar{v}_2\}$ for $\mathbb{R}^2$

where $\bar{v}_1 = (-2,1)$, $\bar{v}_2 = (1,3)$ & let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be the L.T. such that $T(\bar{v}_1) = (-1,2,0)$, $T(\bar{v}_2) = (0,-3,5)$. Find a formula $T(x_1,x_2)$ and use it to compute $T(2,-3)$.

$\Rightarrow$

$$(x_1,x_2) = k_1\bar{v}_1 + k_2\bar{v}_2$$

$$= k_1(-2,1) + k_2(1,3)$$

$$= (-2k_1+k_2,\ k_1+3k_2)$$

$$-2k_1 + k_2 = x_1 \quad \text{--- } 1$$

$$k_1 + 3k_2 = x_2 \quad \text{--- } 2$$

$1 + 2(2)$

$$7k_2 = x_1 + 2x_2$$

$$k_2 = \frac{1}{7}(x_1+2x_2)$$

From 1

$$-2k_1 + \frac{1}{7}(x_1+2x_2) = x_1$$

$$-2k_1 = x_1 - \frac{1}{7}x_1 - \frac{2}{7}x_2$$

$$= \frac{6}{7}x_1 - \frac{2}{7}x_2$$

$$k_1 = \frac{1}{7}(-3x_1+x_2)$$

$$(x_1,x_2) = \frac{1}{7}(-3x_1+x_2)\bar{v}_1 + \frac{1}{7}(x_1+2x_2)\bar{v}_2$$

$$\therefore T(x_1,x_2) = \frac{1}{7}(-3x_1+x_2)T(\bar{v}_1) + \frac{1}{7}(x_1+2x_2)T(\bar{v}_2)$$

$$= \frac{1}{7}(-3x_1+x_2)(-1,2,0) + \frac{1}{7}(x_1+2x_2)(0,-3,5)$$

$$= \frac{1}{7}(3x_1-x_2,\ -6x_1+2x_2-3x_1-6x_2,\ 0+5x_1+10x_2)$$

$$T(x_1,x_2) = \frac{1}{7}(3x_1-x_2,\ -9x_1-4x_2,\ 5x_1+10x_2)$$

$$\therefore T(2,-3) = \frac{1}{7}(6+3,\ -18+12,\ 10-30)$$

$$= \frac{1}{7}(9,-6,-20).$$

Ex. Consider the basis $S = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ for $\mathbb{R}^3$ where $\bar{v}_1 = (-1, 0, 1)$, $\bar{v}_2 = (0, 1, -1)$, $\bar{v}_3 = (1, 1, 1)$ & $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a L.T. s.t. $T(\bar{v}_1) = (1, 0, 0)$, $T(\bar{v}_2) = (0, 1, 0)$, $T(\bar{v}_3) = (0, 0, 1)$. Find a formula $T(a, b, c)$ and use it to compute $T(-1, 2, 9)$.

$\Rightarrow$

$$(a, b, c) = k_1\bar{v}_1 + k_2\bar{v}_2 + k_3\bar{v}_3$$

$$= k_1(-1, 0, 1) + k_2(0, 1, -1) + k_3(1, 1, 1)$$

$$= (-k_1 + k_3,\ k_2 + k_3,\ k_1 - k_2 + k_3)$$

$$\therefore -k_1 + k_3 = a \quad \text{--- (1)}$$

$$k_2 + k_3 = b \quad \text{--- (2)}$$

$$k_1 - k_2 + k_3 = c \quad \text{--- (3)}$$

(2) + (3)

$$k_1 + 2k_3 = b + c \quad \text{--- (4)}$$

(1) + (4)

$$3k_3 = a + b + c$$

$$k_3 = \frac{1}{3}(a + b + c)$$

From (1)

$$-k_1 + \frac{1}{3}(a + b + c) = a$$

$$-k_1 = a - \frac{1}{3}(a + b + c) = +\frac{2}{3}a - \frac{1}{3}b - \frac{1}{3}c$$

$$k_1 = \frac{1}{3}(-2a + b + c)$$

$$k_2 + \frac{1}{3}(a + b + c) = b$$

$$k_2 = b - \frac{1}{3}(a + b + c)$$

$$= -\frac{1}{3}a + \frac{2}{3}b - \frac{1}{3}c$$

$$= \frac{1}{3}(-a + 2b - c)$$

$$(a, b, c) = \frac{1}{3}(-2a + b + c)\bar{v}_1 + \frac{1}{3}(-a + 2b - c)\bar{v}_2 + \frac{1}{3}(a + b + c)\bar{v}_3$$

$$T(a,b,c) = \frac{1}{3}(-2a+b+c)\,T(\bar{v}_1) + \frac{1}{3}(-a+2b-c)\,T(\bar{v}_2) + \frac{1}{3}(a+b+c)\,T(\bar{v}_3)$$

$$= \frac{1}{3}(-2a+b+c)\,(1,0,0) + \frac{1}{3}(-a+2b-c)\,(0,1,0) + \frac{1}{3}(a+b+c)\,(0,0,1)$$

$$T(a,b,c) = \frac{1}{3}(-2a+b+c,\ -a+2b-c,\ a+b+c)$$

$$\therefore\ T(1,-2,9) = \frac{1}{3}(-2-2+9,\ -1-4-9,\ 1-2+9)$$

$$= \frac{1}{3}(5,-14,8)$$

Reference: A textbook of S.Y.B.Sc., Linear Algebra by Golden Series, Nirali Publication.