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Third Year B.Science

V– Sem (2019 CBCS Pattern)
As per the new syllabus

Subject- Classical Mechanics

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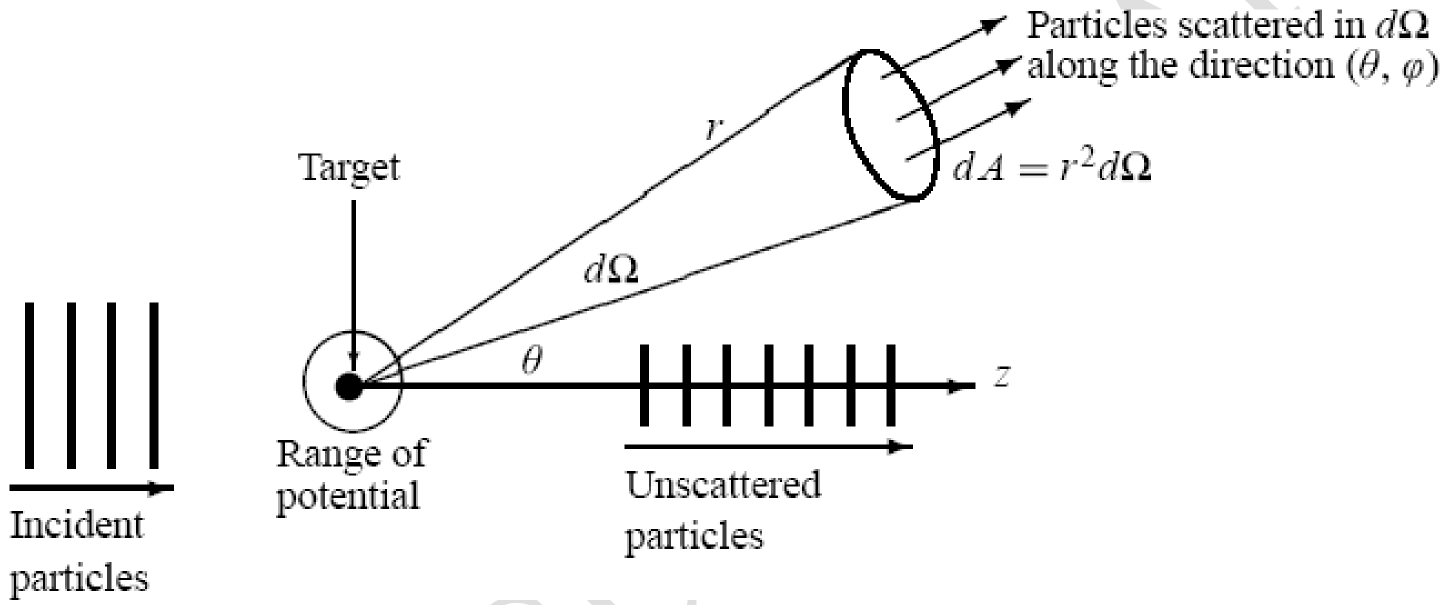
Topic- Scattering of Particles

I. Scattering experiments



Scattering experiment:

A beam of incident scatterers with a given flux or intensity (number of particles per unit area *dA* per unit time *dt*) impinges on the target (described by a scattering potential);

the flux can be written as $J_{\text{inc}} = \frac{dN_{\text{inc}}}{dA\,dt}$

The number of particles per unit time which are detected in a small region of the solid angle, *d*Ω, located at a given angular deflection specified by *(θ, φ)*, can be counted as $\frac{dN_{\text{sc}}}{d\Omega\,dt}$

Scattering cross section

The *differential cross-section* for scattering is defined as the number of particles scattered into an element of solid angle $d\Omega$ in the direction (θ,ϕ) per unit time :

$$\frac{d\sigma}{d\Omega}(\theta,\phi) = \left(\frac{dN_{\rm sc}}{d\Omega dt}\right) \Big/ \left(\frac{dN_{\rm inc}}{dA\, dt}\right) \quad \Rightarrow \quad \frac{d\sigma(\vartheta,\phi)}{d\Omega} = \frac{1}{J_{inc}} \frac{dN_{sc}}{d\Omega} \qquad (1.1)$$

[dimensions of an area]

J_{inc} - incident flux

The *total cross-section* corresponds to scatterings through any scattering angle:

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin(\theta)\, d\theta \frac{d\sigma}{d\Omega}(\theta,\phi) \qquad (1.2)$$

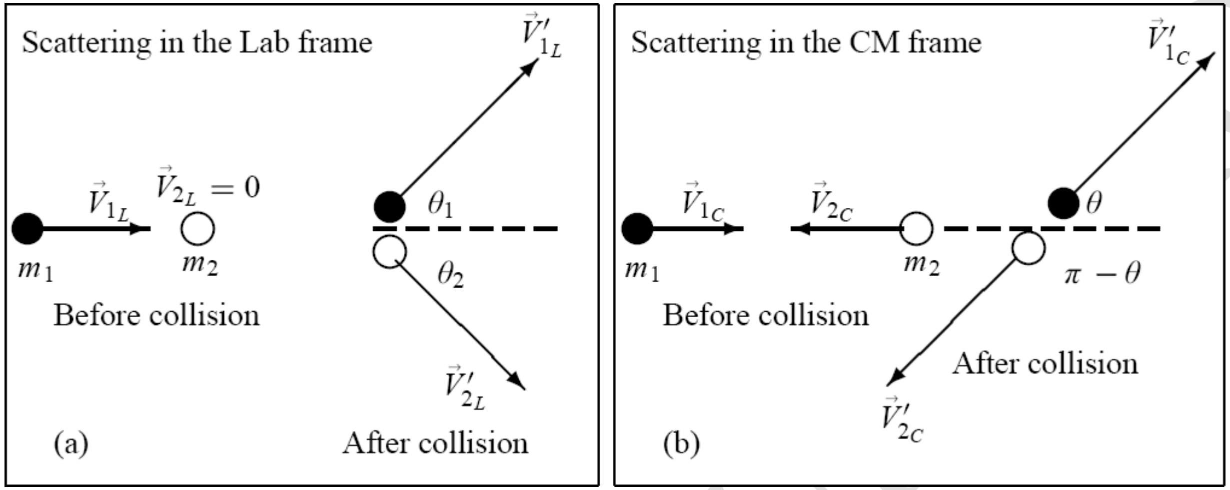
Most scattering experiments are carried out in the laboratory (Lab) frame in which the target is initially at rest while the projectiles are moving. Calculations of the cross sections are generally easier to perform within the center-of-mass (CM) frame in which the center of mass of the projectiles–target system is at rest (before and after collision)

➔ one has to know how to transform the cross sections from one frame into the other.

Note: the total cross section σ is the same in both frames, since the total number of collisions that take place does not depend on the frame in which the observation is carried out. However, the differential cross sections $d\sigma/d\Omega$ are not the same in both frames, since the scattering angles (θ,ϕ) are frame dependent.

Connecting the angles in the Lab and CM frames

Elastic scattering of two structureless particles in the Lab and CM frames:



Note:
consider for simplicity NON-relativistic kinematics!

$$V_{CM} \parallel V_{1L}$$

To find the connection between the Lab and CM cross sections, we need first to find how the scattering angles in one frame are related to their counterparts in the other.

If $\vec{r}_{1_L}$ and $\vec{r}_{1_C}$ **denote the position of** m_1 **in the Lab and CM frames, respectively, and if** $\vec{R}$ **denotes the position of the center of mass with respect to the Lab frame, we have** $\vec{r}_{1_L} = \vec{r}_{1_C} + \vec{R}$. **A time derivative of this relation leads to**

$$\vec{V}_{1_L} = \vec{V}_{1_C} + \vec{V}_{CM}, \qquad (1.3)$$

where $\vec{V}_{1_L}$ and $\vec{V}_{1C}$ **are the velocities of** m_1 **in the Lab and CM frames *before* collision and** $\vec{V}_{CM}$ **is the velocity of the CM with respect to the Lab frame. Similarly, the velocity of** m_1 ***after* collision is**

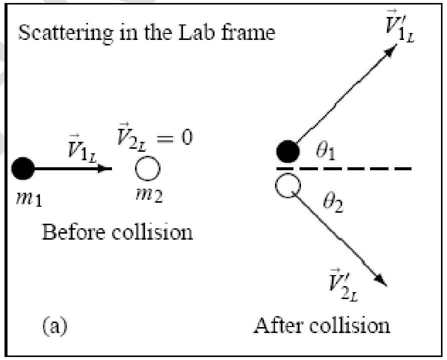
$$\vec{V'}_{1_L} = \vec{V'}_{1_C} + \vec{V}_{CM}. \qquad (1.4)$$

Connecting the angles in the Lab and CM frames

Since $\vec{V}_{CM} \parallel \vec{V}_{1L}$ ⇨

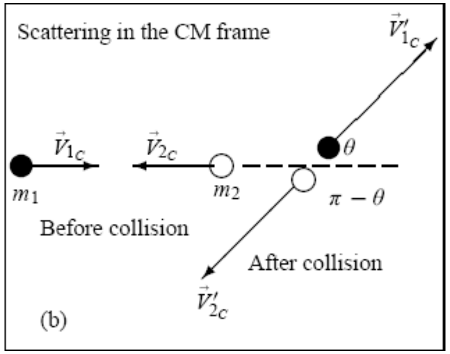
$$V'_{1_L}\cos\theta_1 = V'_{1_C}\cos\theta + V_{CM}, \qquad (1.5)$$

$$V'_{1_L}\sin\theta_1 = V'_{1_C}\sin\theta. \qquad (1.6)$$



Dividing (1.6) by (1.5), we end up with

$$\tan\theta_1 = \frac{\sin\theta}{\cos\theta + V_{CM}/V'_{1_C}}, \qquad (1.7)$$



Use that CM momenta

$$\vec{p}_{CM} = \vec{p}_{1L} + \vec{p}_{2L} \qquad (1.8)$$

$$(m_1 + m_2)\vec{V}_{CM} = m_1\vec{V}_{1L} + m_2\vec{V}_{2L}$$

Since $\vec{V}_{2_L} = 0,$ from (1.8) ➔

$$\vec{V}_{CM} = \frac{m_1\vec{V}_{1_L} + m_2\vec{V}_{2_L}}{m_1+m_2} = \frac{m_1}{m_1+m_2}\vec{V}_{1_L}, \qquad (1.9)$$

or from (1.9) ➔

$$\vec{V}_{1_L} = \vec{V}_{1_C} + m_1\vec{V}_{1_L}/(m_1+m_2); \qquad (1.10)$$

$$\vec{V}_{1_C} = \left(1 - \frac{m_1}{m_1+m_2}\right)\vec{V}_{1_L} = \frac{m_2}{m_1+m_2}\vec{V}_{1_L}. \qquad (1.11)$$

Connecting the angles in the Lab and CM frames

Since the center of mass is at rest in the CM frame, the total momenta before and after collisions are separately zero: $\boldsymbol{p}_C = \boldsymbol{p}_{1C} + \boldsymbol{p}_{2C} = \vec{\boldsymbol{p}}'_C = \boldsymbol{p}'_{1C} + \boldsymbol{p}'_{2C} = \boldsymbol{0}$

$$p_C = m_1 V_{1_C} - m_2 V_{2_C} = 0 \implies V_{2_C} = \frac{m_1}{m_2} V_{1_C}, \qquad (1.12)$$

$$p'_C = m_1 V'_{1_C} \cos\theta - m_2 V'_{2_C} \cos\theta = 0 \implies V'_{2_C} = \frac{m_1}{m_2} V'_{1_C}. \qquad (1.13)$$

Since the kinetic energy is conserved:

$$\tfrac{1}{2} m_1 V_{1_C}^2 + \tfrac{1}{2} m_2 V_{2_C}^2 = \tfrac{1}{2} m_1 {V'_{1_C}}^2 + \tfrac{1}{2} m_2 {V'_{2_C}}^2 \qquad (1.14)$$

$$\boxed{V'_{1_C} = V_{1_C} \text{ and } V'_{2_C} = V_{2_C}.} \qquad (1.15)$$

In the case of *elastic* collisions, the speeds of the particles in the CM frame are the same before and after the collision;

From (1.11) ➔

$$\vec{V}'_{1_C} = \vec{V}_{1_C} = \frac{m_2}{m_1 + m_2} \vec{V}_{1_L}. \qquad (1.16)$$

Dividing (1.9) by (1.16) ➔

$$\boxed{\frac{V_{CM}}{V'_{1_C}} = \frac{m_1}{m_2}.} \qquad (1.17)$$

Connecting the angles in the Lab and CM frames

Finally, a substitution of (1.17) into (1.7) yields

$$\tan\theta_1 = \frac{\sin\theta}{\cos\theta + V_{2C}/V_{1C}} = \frac{\sin\theta}{\cos\theta + m_1/m_2}, \qquad (1.17)$$

and using $\cos\theta_1 = 1/\sqrt{\tan^2\theta_1 + 1}$,
we obtain

$$\cos\theta_1 = \frac{\cos\theta + \frac{m_1}{m_2}}{\sqrt{1 + \frac{m_1^2}{m_2^2} + 2\frac{m_1}{m_2}\cos\theta}}. \qquad (1.18)$$

Note: **In a similar way we can establish a connection between θ_2 and θ.**

From (1.4) we have $\vec{V}'_{2_L} = \vec{V}'_{2_C} + \vec{V}_{CM}$. **+ using** $V_{CM} = V'_{2_C} = V_{2C}$.

➔ the *x* and *y* components of this relation are

$$V'_{2_L}\cos\theta_2 = -V'_{2_C}\cos\theta + V_{CM} = (-\cos\theta + 1)V'_{2_C}, \qquad (1.19)$$

$$V'_{2_L}\sin\theta_2 = -V'_{2_C}\sin\theta;$$

$$\tan\theta_2 = \frac{\sin\theta}{-\cos\theta + V_{CM}/V'_{2_C}} = \frac{\sin\theta}{1 - \cos\theta} = \cot\left(\frac{\theta}{2}\right) \Longrightarrow \theta_2 = \frac{\pi - \theta}{2}. \qquad (1.20)$$

Connecting the Lab and CM Cross Sections

The connection between the differential cross sections in the Lab and CM frames can be obtained from the fact that the number of scattered particles passing through an infinitesimal cross section is the same in both frames:

$$d\sigma(\theta_1, \varphi_1) = d\sigma(\theta, \varphi) \qquad (1.21)$$

What differs is the solid angle $d\Omega$:

in the Lab frame: $d\Omega_1 = \sin\theta_1 d\theta_1 d\varphi_1$

in the CM frame: $d\Omega = \sin\theta\, d\theta d\varphi.$

$$\left(\frac{d\sigma}{d\Omega_1}\right)_{Lab} d\Omega_1 = \left(\frac{d\sigma}{d\Omega}\right)_{CM} d\Omega \Longrightarrow \left(\frac{d\sigma}{d\Omega_1}\right)_{Lab} = \left(\frac{d\sigma}{d\Omega}\right)_{CM} \frac{\sin\theta}{\sin\theta_1}\frac{d\theta}{d\theta_1}\frac{d\varphi}{d\varphi_1}, \qquad (1.22)$$

Since there is a cylindrical symmetry around the direction of the incident beam $\varphi = \varphi_1$

$$\Rightarrow \quad \left(\frac{d\sigma}{d\Omega_1}\right)_{Lab} = \left(\frac{d\sigma}{d\Omega}\right)_{CM} \frac{d(\cos\theta)}{d(\cos\theta_1)}. \qquad (1.23)$$

From (1.18) ➔

$$\frac{d\cos\theta_1}{d\cos\theta} = \frac{1 + \frac{m_1}{m_2}\cos\theta}{\left(1 + \frac{m_1^2}{m_2^2} + 2\frac{m_1}{m_2}\cos\theta\right)^{3/2}} \qquad (1.24)$$

Connecting the Lab and CM Cross Sections

Thus :

$$\left(\frac{d\sigma}{d\Omega_1}\right)_{Lab} = \frac{\left(1+\frac{m_1^2}{m_2^2}+2\frac{m_1}{m_2}\cos\theta\right)^{3/2}}{1+\frac{m_1}{m_2}\cos\theta}\left(\frac{d\sigma}{d\Omega}\right)_{CM} \quad (1.25)$$

From (1.23) and (1.20) ➔

$$\left(\frac{d\sigma}{d\Omega_2}\right)_{Lab} = 4\cos\theta_2\left(\frac{d\sigma}{d\Omega_2}\right)_{CM} = 4\sin\left(\frac{\theta}{2}\right)\left(\frac{d\sigma}{d\Omega_2}\right)_{CM} \quad (1.26)$$

Limiting cases:

1) If $m_2 \gg m_1$, or when $\frac{m_1}{m_2} \to 0$, **the Lab and CM results are the same,**

since (1.17) leads to $\theta_1=\theta$ **➔** $\left(\frac{d\sigma}{d\Omega_1}\right)_{Lab} = \left(\frac{d\sigma}{d\Omega}\right)_{CM}$ (1.27)

2) If $m_2 = m_1$ **then from (1.20) ➔** $\tan\theta_1 = \tan(\theta/2)$ or to $\theta_1 = \theta/2$;

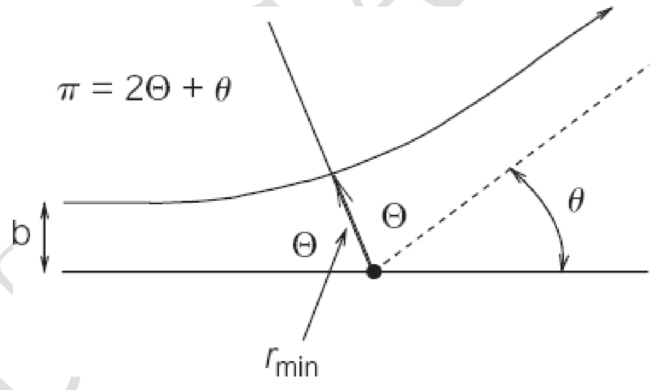
from (1.25) → $\left(\frac{d\sigma}{d\Omega_1}\right)_{Lab} = 4\left(\frac{d\sigma}{d\Omega}\right)_{CM}\cos(\theta/2).$ (1.28)

From classical to quantum scattering theory

1. **Classical theory:**

 - **scattering of hard ‚spheres‘**

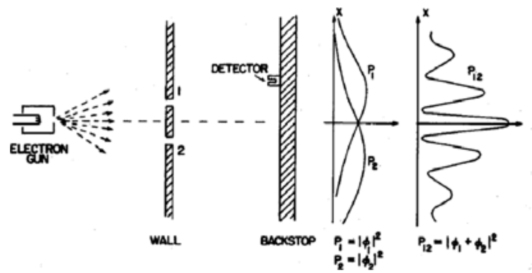
 - **individual well-defined trajectories**

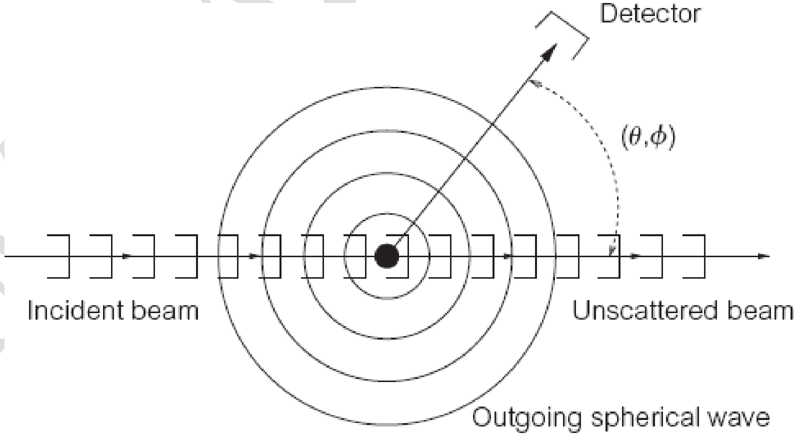


2. **Quantum theory:**

 - **scattering of wave pakeges ← wave–particle duality**

 - **probabilistic origin of scattering process**

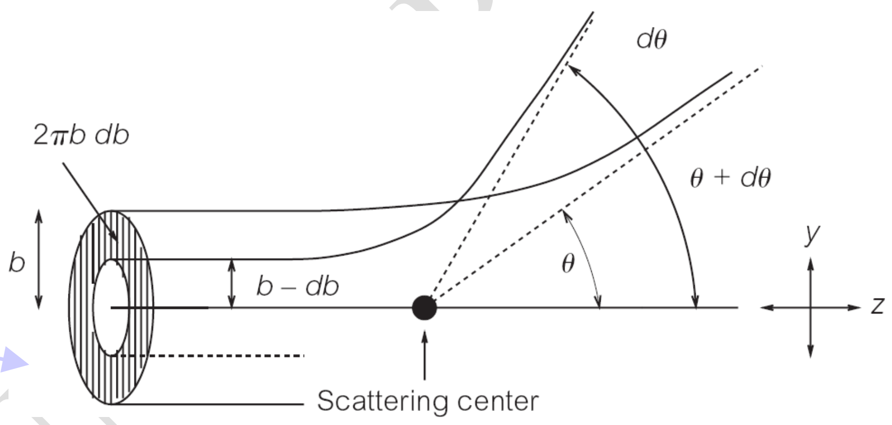




II. Classical trajectories and cross-sections

Scattering trajectories, corresponding to different impact parameters, *b* give different scattering angles *θ*.

All of the particles in the beam in the hatched region of area $d\sigma = 2\pi\, b db$ are scattered into the angular region $(\theta, \theta + d\theta)$



The equations of motion for incident particle:

$$m\ddot{\mathrm{r}}(t) = \mathrm{F}(r) \quad (2.1)$$

+ initial conditions:

$$y(t=-\infty) = b \quad \dot{y}(t=-\infty) = 0 \quad (2.2)$$

$$z(t=-\infty) = -\infty \quad v_\infty \equiv \dot{z}(t=-\infty) = \sqrt{2mE} \quad (2.3)$$

➔ defines the trajectory

since initial kinetic energy $E = mv_\infty^2/2$

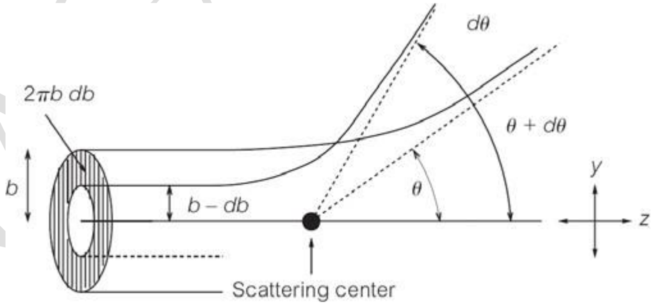
For a particle obeying classical mechanics:

- **the trajectory for the unbound motion, corresponding to a scattering event, is deterministically predictable, given by the interaction potential and the initial conditions;**
- **the path of any scatterer in the incident beam can be followed, and its angular deflection is determined as precisely as required**

The number of particles scattered per unit time into the angular region *(θ, θ+dθ)* with any value of *φ* can be written as

(2.4) $$\frac{dN_{sc}}{dt} = d\sigma \frac{dN_{inc}}{dA\, dt} \quad \text{where} \quad d\sigma = 2\pi\, b\, db$$

$$\frac{d\sigma}{d\theta} = 2\pi\, b(\theta) \left| \frac{db(\theta)}{d\theta} \right|$$



(2.5) $$\frac{d\sigma}{d\Omega} = \left[\frac{1}{2\pi\, \sin(\theta)} \right] 2\pi\, b(\theta) \left| \frac{db(\theta)}{d\theta} \right| = \frac{b(\theta)}{\sin(\theta)} \left| \frac{db(\theta)}{d\theta} \right|$$

$$d\Omega = 2\pi\, \sin(\theta) d\theta$$

The knowledge of *b(θ)*, obtained directly from Newton's laws or other methods, is then sufficient to calculate the scattering cross-section.

For the scattering from nontrivial central forces, the trajectory can be obtained from the equations of motion by using energy- and angular momentum- conservation methods

E.g., one can rewrite $$E = \frac{1}{2} m \dot{r}^2 + \frac{L^2}{2mr^2} + V(r)$$ **L - angular momentum**

in the form $$\sqrt{\frac{2}{m}\left(E - \frac{L^2}{2mr^2} - V(r)\right)} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\dot{\theta}$$ (2.6)

The angular momentum can be written via the angular velocity $L = mr^2\dot{\theta}$

➔from (2.6)

$$d\theta = \left(\frac{L}{r^2\sqrt{2m(E - L^2/2mr^2 - V(r))}}\right) dr \qquad (2.7)$$

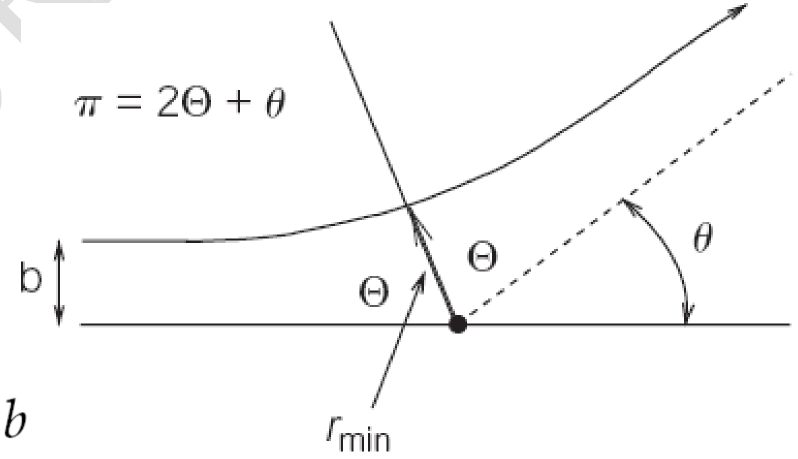
➔ the angle through which the particle moves between two radial distances r_1 and r_2 :

$$\Delta\theta = \int_{r_1}^{r_2} dr \frac{L}{r^2\sqrt{2m(E - L^2/2mr^2 - V(r))}} \qquad (2.8)$$

Scattering trajectories in a central potential :

r_{min} - the distance of closest approach,

Θ – deflection angle



(2.9) $$2\Theta + \theta = \pi \quad \text{or} \quad \Theta = \frac{\pi}{2} - \frac{\theta}{2}$$

Use that the initial angular momentum $L = mv_\infty b$ and $E = mv_\infty^2/2$ ➔ $L = b\sqrt{2mE}$

➔ from (2.8), (2.9)

$$\Theta = \int_{r_{min}}^{\infty} dr \frac{b}{r^2\sqrt{1 - b^2/r^2 - V(r)/E}} \qquad (2.10)$$

❖ Impact Parameter

In a real scattering experiment, information about the scatterer can be figured out from the different rates of scattering to different angles. Detectors are placed at various angles (θ,ϕ). Of course, a physical detector collects scattered particles over some nonzero solid angle.

The usual notation for infinitesimal solid angle is $d\Omega=\sin\theta d\theta d\phi$. 1

The full solid angle (all possible scatterings) is $\int d\Omega=4\pi$ the area of a sphere of unit radius.

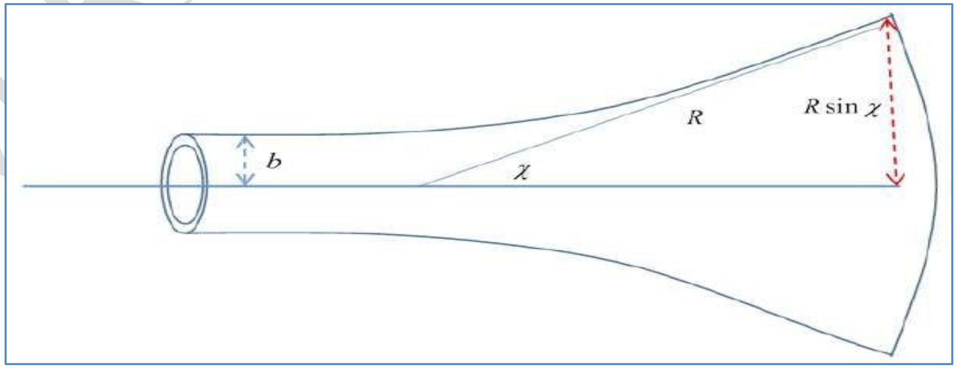
(*Note*: Landau uses do for solid angle increment, but $d\Omega$ has become standard.)

The differential cross section, written $d\sigma/d\Omega$ is the fraction of the total number of scattered particles that come out in the solid angle $d\Omega$, so the rate of particle scattering to this detector is $nd\sigma/d\Omega$, with n the beam intensity as defined above.

Now, we'll assume the potential is spherically symmetric. Imagine a line parallel to the incoming particles going through the center of the atom. For a given ingoing particle, its **impact parameter** is defined as the distance its ingoing line of flight is from this central line. Landau calls this ρ, we'll follow modern usage and call it b.

A particle coming in with impact parameter between b and $b+db$ will be scattered through an angle between χ and $\chi+d\chi$ where we're going to calculate, $\chi(b)$ by solving the equation of motion of a single particle in a repulsive inverse-square force.

Note: we've switched for this occasion from θ to χ for the angle scattered through because we want to save θ for the (r,θ) coordinates describing the complete trajectory, or orbit, of the scattered particle.



So, an ingoing cross section $d\sigma = 2\pi b db$ 2
scatters particles into an outgoing spherical area (centered on the scatterer) $2\pi R \sin\chi R d\chi$ 3
that is, a solid angle

$$d\Omega = 2\pi \sin\chi d\chi \qquad 4$$

Therefore the scattering *differential cross section*

$$d\sigma/d\Omega = (b(\chi) / \sin\chi) * |db/d\chi|$$

(Note that $d\chi/db$dχ/db is clearly negative—increasing b means increasing distance from the scatterer, so a smaller χχ)

- Total cross –section:

The total cross-section can be obtained by using equation, we get,

$$\sigma = \int \boldsymbol{\sigma}(\Omega) \boldsymbol{d}\Omega$$

$$\sigma = 2\pi \int \boldsymbol{\sigma}\, \boldsymbol{\Theta} \boldsymbol{sin\Theta d\Theta}$$