K.T.S.P. Mandal's

HutatmaRajguruMahavidyalaya
Rajgurnagar-410505

Third Year B.Science

V– Sem (2019 CBCS Pattern)
As per the new syllabus

Subject- Classical Electrodynamics

Mr. N.D.Barne
Department of Physics
nileshbarne11@gmail.com

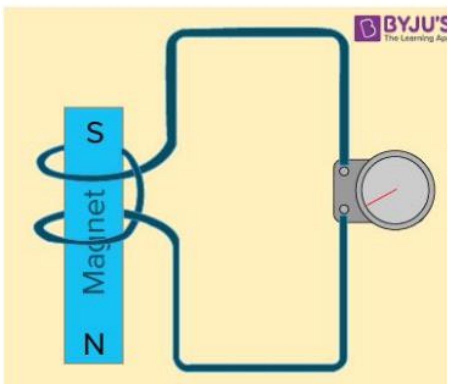
Topic- Electrodynamics

❖ Concept of Electromagnetic Induction:

Electromagnetic Induction was **discovered by Michael Faraday** in 1831 and James Clerk Maxwell mathematically described it as Faraday's law of induction.

Electromagnetic Induction is a current produced because of voltage production (electromotive force) due to a changing magnetic field.
This either happens when a conductor is placed in a moving magnetic field (when using AC power source) or when a conductor is constantly moving in a stationary magnetic field.

Michael Faraday arranged a conducting wire as per the setup given below, attached to a device to measure the voltage across the circuit. When a bar magnet was moved through the coiling, the voltage detector measures the voltage in the circuit.



Through his experiment, he discovered that there are certain factors that influence this voltage production. They are:

1. ***Number of Coils***: The induced voltage is directly proportional to the number of turns/coils of the wire. Greater the number of turns, greater is voltage produced
2. ***Changing Magnetic Field***: Changing magnetic field affects the induced voltage. This can be done by either moving the magnetic field around the conductor or moving the conductor in the magnetic field.

Applications of Electromagnetic Induction

Based on his experiments we now have *Faraday's law* according to which *the amount of voltage induced in a coil is proportional to the number of turns of the coil and the rate of changing magnetic field.*

- AC generator works on the principle of electromagnetic induction.
- The working of electrical transformers are based on the electromagnetic induction.
- The magnetic flow meter is based on the electromagnetic induction.

Faraday’s Law of Induction and Lenz’s Law:

Considering two neighboring circuits as shown in adjacent figure, one circuit connected with a battery and a key and other circuit contains only a galvanometer. When key K is pressed a deflection is observed in the galvanometer G. when K is kept pressed then deflection in G becomes zero. When suddenly key K is released then current in first circuit now becomes zero and due to this there is again a deflection is observed in galvanometer G but this time in opposite direction as earlier. The deflection in galvanometer is also observed if the battery circuit is held stationary and the galvanometer circuit is moved and vice-versa. Therefore deflection in galvanometer G is observed whenever the two circuits are in relative motion.

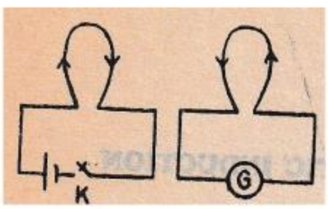


Fig 1

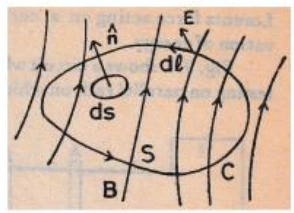


Fig 2

Faraday concluded from the above experiment that “whenever there is change in the magnetic flux in one circuit then an induced e.m.f and hence induced current (in closed circuit) is produced in the neighboring circuit, the magnitude of this induced current depends upon the time rate of change of magnetic flux. The changing magnetic flux produces an induced emf ξ which is represented by:

$$\xi = \oint E.\,dl \;\ldots\ldots\ldots\ldots\ldots(1)$$

E.M.F may be defined as the work done by the electric field E in carrying a unit positive charge in the complete circuit. let us consider a circuit C enclosing a surface S as shown in fig 2. The magnetic field produced in the neighboring circuit is B. Consider a small surface element dS of the surface enclosed by the circuit. then the magnetic flux linked by the circuit C will be given by:

$$\phi = \int_s B.\hat{n}\, dS \;\ldots\ldots\ldots\ldots\ldots(2)$$

Where n ^ is a unit vector along the positive normal to dS.

The Faraday's law of electromagnetic induction states that “the line integral of the electric field E around the circuit is equal to the negative of the time rate of change of the total magnetic flux through the circuit. The sign of the e.m.f is specified by Lenz’s law.

Therefore

$$\oint E.dl = -\frac{\partial\phi}{\partial t} = -\frac{\partial}{\partial t}[\int_s B.\hat{n}dS] \text{ ....(3)}$$

Equation (3) represents the ***Integral form*** of Faraday's law of electromagnetic induction.

When the circuit remains fixed and B changes then the derivative $\frac{\partial}{\partial t}[\int_s B.\hat{n}dS]$ doesn't depend on path C and so $\frac{\partial}{\partial t}$ can be taken inside the integral.

$$\oint E.dl = -[\int_s \frac{\partial B}{\partial t}.\hat{n}dS] \text{ ....(4)}$$

According to curl stokes theorem;

$$\oint E.dl = \int_s \nabla \times E.\hat{n}dS \text{ ....(5)}$$

From (4) and (5); $\int_s \nabla \times E.\hat{n}dS = -[\int_s \frac{\partial B}{\partial t}.\hat{n}dS]$

$$\boldsymbol{\nabla \times E = -\frac{\partial B}{\partial t}} \text{ ....(6)}$$

This is the Faraday's law of electromagnetic induction expressed in ***differential form*** and is the time-dependent generalisation of the statement of $\nabla \times E = 0$ for electrostatic fields.

Lenz's Law

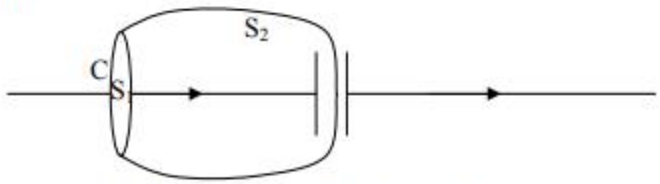
Lenz's law gives the direction of induced current. It states that direction of induced current in a closed circuit is such that it opposes the very cause that produces it, that is, the induced current produces a magnetic field which opposes the change in flux. Thus if the flux ϕ increases, the current produces an opposing flux and if flux decreases the induced current produces an aiding flux.

The law applies only to the closed circuits. This law follows from the principle of conservation of energy.

According to Lenz's law, the direction of the current should be such as to oppose the motion of the magnet towards the circuit. Now we know that a current through a coil produces magnetic field like a magnetic dipole. One face of the coil, therefore behaves as North Pole of a magnet and the other as South Pole. The face from which the lines of magnetic field emerge will obviously act as North Pole and vice versa.

❖ Displacement Current and Generalization of Ampere's Law:

We have seen that Ampere's law $\oint_C \vec{B}\cdot d\vec{s} = \mu_o I_{enclosed}$ allows us to calculate B in those cases of sufficient symmetry, in analogy with Gauss's law allowing us to find E in high symmetry situations. We mentioned that in the form above Ampere's law is limited to constant currents, or equivalently, to constant electric fields in the wires producing those constant currents. About 140 years ago, Maxwell generalized this to allow for time-varying currents, or electric fields.



To see the modification to Ampere's law in the form above when there are time-varying electric fields, consider an electric circuit with a capacitor in it. If we analyze Ampere's law, in the form above, using a closed loop around the wire with current I flowing into the capacitor (as shown), we would set the contour integral equal to $\mu_o I$. However, the enclosed current does not continue between the capacitor plates – there is no "conduction current" in the gap between these plates. The enclosed current is really defined as the integral of the current density over a surface S: $I_{enclosed} = \iint_S \vec{J}\cdot d\vec{A}$,

or the flux of the current density. The surface S is any surface bounded by the curve C used in the contour integral. Here's the dilemma. If we use as a surface S_1, the circular plane area that is bounded by C, then the current enclosed will just be I, the current in the wire. If instead we use the surface S_2, also bounded by C, but going between the capacitor plates, then the conduction current crossing this surface is zero – since there is no conduction current in the gap between the capacitor plates – and therefore the $I_{enclosed}$ is zero and we have a contradiction. Maxwell recognized this contradiction and introduced an extra current – called a **displacement current** – I_d, defined as

$$I_d = \varepsilon_o \frac{d\Phi_E}{dt},$$

where Φ_E is the electric field flux (displacement here has nothing to do with a vector distance). As the capacitor is being charged or discharged and its Q is changing, there will be a changing E field within the gap between the capacitor plates and therefore there will be a displacement current there. If we generalize Ampere's law to include not only conduction currents, but also displacement currents, then the above contradiction disappears. So, we have the generalization of Ampere's law to be

$$\oint_C \vec{B}\cdot d\vec{s} = \mu_o(I + I_d) = \mu_o I + \mu_o\varepsilon_o \frac{d\Phi_E}{dt}.$$

For the case of the capacitor let's evaluate I_d. Since $E = \sigma/\varepsilon_o$ between the capacitor plates, We can compute $\Phi_E = EA = \sigma A/\varepsilon_o = Q/\varepsilon_o$, where Q is the charge on the capacitor. Then $I_d = dQ/dt = I$, and we get the same result for the line integral of B whether we use surface S_1 or S_2.

❖ Maxwell's Equations (Differential Form and Integral Forms) and Their Physical Significance:

1) Differential Form:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \vec{M}_i = -\vec{M}_d - \vec{M}_i$$

$$\nabla \times \vec{H} = \vec{J}_i + \vec{J}_c + \frac{\partial \vec{D}}{\partial t} = \vec{J}_i + \vec{J}_c + \vec{J}_d$$

$$\nabla \cdot \vec{D} = \rho_{ev}$$

$$\nabla \cdot \vec{B} = \rho_{mv}$$

$$\vec{M}_d = \frac{\partial \vec{B}}{\partial t}, \qquad \vec{J}_d = \frac{\partial \vec{D}}{\partial t}$$

$\vec{E} \equiv$ Electric field intensity [V/m]

$\vec{B} \equiv$ Magnetic flux density [Weber/m^2 = V s/m^2 = Tesla]

$\vec{M}_i \equiv$ Impressed (source) magnetic current density [V/m^2]

$\vec{M}_d \equiv$ Magnetic displacement current density [V/m^2]

$\vec{H} \equiv$ Magnetic field intensity [A/m]

$\vec{J}_i \equiv$ Impressed (source) electric current density [A/m^2]

$\vec{D} \equiv$ Electric flux density or electric displacement [C/m^2]

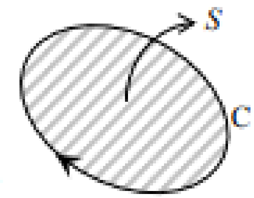
$\vec{J}_c \equiv$ Electric conduction current density [A/m^2]

$\vec{J}_d \equiv$ Electric displacement current density [A/m^2]

$\rho_{ev} \equiv$ Electric charge volume density [C/m^3]

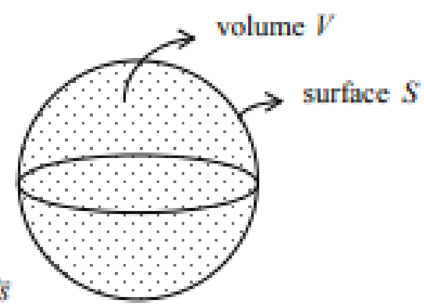
2) Integral Form:

Elementary vector calculus:



Stokes' Theorem: $\iint_S (\nabla \times \vec{A}) \cdot d\vec{s} = \oint_C \vec{A} \cdot d\vec{l}$

• It says that if you want to know what is happening in the interior of a surface bounded by a curve just go around the curve and add up the field contributions.



Divergence Theorem: $\iiint_V (\nabla \cdot \vec{A})\, dV = \oiint_S \vec{A} \cdot d\vec{s}$

• In simple words, divergence theorem states that if you want to know what is happening within a volume of V just go around the surface S (bounding volume V) and add up the field contributions.

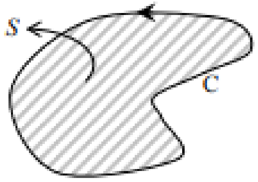
Null Identities:

$\nabla \cdot (\nabla \times \vec{A}) = 0 \Leftrightarrow \nabla \cdot \vec{B} = 0 \Leftrightarrow \vec{B} = \nabla \times \vec{A}$

$\nabla \times (\nabla \phi) = 0 \Leftrightarrow \underset{\text{(electrostatic)}}{\nabla \times \vec{E} = 0} \Leftrightarrow \vec{E} = -\nabla \phi$

• The Divergence and Stokes' theorems can be used to obtain the integral forms of the Maxwell's Equations from their differential form.

• $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \vec{M}_i \Rightarrow \iint_S \nabla \times \vec{E} \cdot d\vec{s} = -\frac{\partial}{\partial t} \iint_S \vec{B} \cdot d\vec{s} - \iint_S \vec{M}_i \cdot d\vec{s}$



$$\Rightarrow \oint_C \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\iint_S \vec{B}\cdot d\vec{s} - \iint_S \vec{M}_i \cdot d\vec{s}$$

- $$\nabla\times\vec{H} = \frac{\partial\vec{D}}{\partial t} + \sigma\vec{E} + \vec{J}_i \Leftrightarrow \iint_S \nabla\times\vec{H}\cdot d\vec{s} = \iint_S \frac{\partial\vec{D}}{\partial t}\cdot d\vec{s} + \iint_S \sigma\vec{E}\cdot d\vec{s} + \iint_S \vec{J}_i\cdot d\vec{s}$$

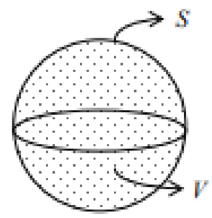
$$\Rightarrow \oint_C \vec{H}\cdot d\vec{l} = \frac{\partial}{\partial t}\iint_S \vec{D}\cdot d\vec{s} + \iint_S \vec{J}_c\cdot d\vec{s} + \iint_S \vec{J}_i\cdot d\vec{s}$$

Where $\vec{J}_c = \sigma\vec{E}$

- $$\nabla\cdot\vec{D} = \rho_{ev} \Leftrightarrow \iiint_V \nabla\cdot\vec{D}\,dv = \iiint_V \rho_{ev}\,dv \Rightarrow \oint\!\!\oint_S \vec{D}\cdot d\vec{s} = \iiint_V \rho_{ev}\,dv = Q_e$$

- $$\nabla\cdot\vec{B} = \rho_{mv} \Leftrightarrow \iiint_V \nabla\cdot\vec{B}dv = \iiint_V \rho_{mv}dv \Rightarrow \oint\!\!\oint_S \vec{B}\cdot d\vec{s} = Q_m = \iiint_V \rho_{mv}dv$$

Since $Q_m = 0 \Rightarrow \oint\!\!\oint_S \vec{B}\cdot d\vec{s} = 0$



❖ Reflection and refraction of plane waves at an interface:

Consider space filled with two uniform linear materials, one with permitivity and permeability, ϵ, μ and the other with ϵ', μ'. Let the first occupy all space below the xy-plane and the other all space above the xy-plane. The usual electromagnetic boundary conditions therefore apply at the interface at $z = 0$.

Let the incident wave move with wave vector, $\mathbf{k}$, frequency ω, and fields $\mathbf{E}, \mathbf{B}$. Denote the same quantities for the refracted wave by $\mathbf{k}', \omega', \mathbf{E}', \mathbf{B}'$, and those for the reflected wave by $\mathbf{k}'', \omega'', \mathbf{E}'', \mathbf{B}''$.

In order to avoid buildup of energy at the boundary, the space and time dependence of the incoming, reflected, and refracted waves must match,

$$\begin{aligned} e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\Big|_{z=0} &= e^{i(\mathbf{k}'\cdot\mathbf{x}-\omega' t)}\Big|_{z=0} = e^{i(\mathbf{k}''\cdot\mathbf{x}-\omega'' t)}\Big|_{z=0} \\ \mathbf{k}\cdot\mathbf{x}-\omega t &= \mathbf{k}'\cdot\mathbf{x}-\omega' t = \mathbf{k}''\cdot\mathbf{x}-\omega'' t \end{aligned}$$

These relations have to hold at all times and at all x and y. Therefore,

$$\omega = \omega' = \omega''$$

and

$$k_x x + k_y y = k_x' x + k_y' y = k_x'' x + k_y'' y$$

We choose the incoming wave to lie in the xz-plane, so that $k_y = 0$. Let the incoming wave make and angle of incidence, i, with the normal to the interface; let the refracted wave make an acute angle r and the reflected wave an acute angle r' with the normal. Varying x and y independently, it immediately follows that $k_y' = 0$ and $k_y'' = 0$, so the three waves are coplanar. For the x components, we have

$$k\sin i = k'\sin r = k''\sin r'$$

Since the incident and reflected waves are in the same medium, $k = k''$, and therefore the angle of reflection equals the angle of incidence,

$$r' = i$$

while, using the relationship for index of refraction, $n = \frac{c}{v} = c\sqrt{\mu\epsilon} = \frac{ck}{\omega}$, we have

$$\frac{n'}{n} = \frac{k'}{k}$$

and therefore, Snell's law for the angle of refraction,

$$n \sin i = n' \sin r$$

Now we turn to the boundary conditions. With $\mathbf{n}$ the unit normal to the interface (that is, a unit vector in the positive z direction), we have:

1. Continuity of the normal component of $\mathbf{D}$,

$$\epsilon\left(\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}''\right) \cdot \mathbf{n} = \epsilon' \boldsymbol{\mathcal{E}}' \cdot \mathbf{n}$$

2. Continuity of the tangential component of $\mathbf{E}$,

$$\left(\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}''\right) \times \mathbf{n} = \boldsymbol{\mathcal{E}}' \times \mathbf{n}$$

3. Continuity of the normal component of $\mathbf{B}$. Using $\boldsymbol{\mathcal{B}} = \frac{1}{k}\sqrt{\mu\epsilon}\mathbf{k} \times \boldsymbol{\mathcal{E}} = \frac{1}{\omega}\mathbf{k} \times \boldsymbol{\mathcal{E}}$ and cancelling the overall ω,

$$\begin{aligned}\left(\boldsymbol{\mathcal{B}} + \boldsymbol{\mathcal{B}}''\right) \cdot \mathbf{n} &= \boldsymbol{\mathcal{B}}' \cdot \mathbf{n} \\ \left(\mathbf{k} \times \boldsymbol{\mathcal{E}} + \mathbf{k}'' \times \boldsymbol{\mathcal{E}}''\right) \cdot \mathbf{n} &= \mathbf{k}' \times \boldsymbol{\mathcal{E}}' \cdot \mathbf{n}\end{aligned}$$

Continuity of the tangential component of H

$$\frac{1}{\mu}\left(\mathbf{k} \times \boldsymbol{\mathcal{E}} + \mathbf{k}'' \times \boldsymbol{\mathcal{E}}''\right) \times \mathbf{n} = \frac{1}{\mu'}\left(\mathbf{k}' \times \boldsymbol{\mathcal{E}}'\right) \times \mathbf{n}$$

Instead of applying these equations to a general incident polarization, we may treat the two independent linear polarization directions independently, with the general result being a linear combination of the two. We therefore need consider only two particular cases, (1) the electric field perpendicular to the plane incidence (i.e., the +y-direction), and (2) parallel to the plane of incidence.

Polarization perpendicular to the plane of incidence:

Let the electric field vector, $\boldsymbol{\mathcal{E}}$, point in the positive y direction. Setting

$$\begin{aligned}\boldsymbol{\mathcal{E}} &= \mathbf{j}\mathcal{E} \\ \boldsymbol{\mathcal{E}}' &= \mathbf{j}\mathcal{E}' \\ \boldsymbol{\mathcal{E}}'' &= \mathbf{j}\mathcal{E}''\end{aligned}$$

the first boundary condition vanishes identically, while the remaining boundary csonditions become (if you need more detail on this, look below at the second case),

$$\begin{aligned}\mathcal{E} + \mathcal{E}'' &= \mathcal{E}' \\ \left(k\mathcal{E} + k''\mathcal{E}''\right)\sin i &= k'\mathcal{E}' \sin r \\ \frac{1}{\mu}\left(k\mathcal{E} - k''\mathcal{E}''\right)\cos i &= \frac{1}{\mu'}k'\mathcal{E}' \cos r\end{aligned}$$

Using Snell's law in the form $k \sin i = k' \sin r$, the first two of these are identical, while using

$$\frac{k}{\mu} = \omega\sqrt{\frac{\epsilon}{\mu}}$$

to express the result in terms of properties of the medium, then cancelling an overall factor of ω, the third becomes

$$\sqrt{\frac{\epsilon}{\mu}}\left(\mathcal{E} - \mathcal{E}''\right)\cos i = \sqrt{\frac{\epsilon'}{\mu'}}\mathcal{E}'\cos r$$

Eliminating $\mathcal{E}'$ using the first equation, we solve for $\frac{\mathcal{E}''}{\mathcal{E}}$,

$$\begin{aligned}
\sqrt{\frac{\epsilon}{\mu}}\left(\mathcal{E} - \mathcal{E}''\right)\cos i &= \sqrt{\frac{\epsilon'}{\mu'}}\left(\mathcal{E} + \mathcal{E}''\right)\cos r \\
\sqrt{\frac{\epsilon}{\mu}}\mathcal{E}\cos i - \sqrt{\frac{\epsilon}{\mu}}\mathcal{E}''\cos i &= \sqrt{\frac{\epsilon'}{\mu'}}\mathcal{E}\sqrt{1-\sin^2 r} + \sqrt{\frac{\epsilon'}{\mu'}}\mathcal{E}''\sqrt{1-\sin^2 r} \\
\left(\sqrt{\frac{\epsilon}{\mu}}\cos i - \sqrt{\frac{\epsilon'}{\mu'}}\sqrt{1-\sin^2 r}\right)\mathcal{E} &= \left(\sqrt{\frac{\epsilon}{\mu}}\cos i + \sqrt{\frac{\epsilon'}{\mu'}}\sqrt{1-\sin^2 r}\right)\mathcal{E}''
\end{aligned}$$

so that

$$\frac{\mathcal{E}''}{\mathcal{E}} = \frac{\sqrt{\frac{\epsilon}{\mu}}\cos i - \sqrt{\frac{\epsilon'}{\mu'}}\sqrt{1-\sin^2 r}}{\sqrt{\frac{\epsilon}{\mu}}\cos i + \sqrt{\frac{\epsilon'}{\mu'}}\sqrt{1-\sin^2 r}}$$

Next we want to express this in terms of the index of refraction,

$$n = \frac{c}{v} = c\sqrt{\mu\epsilon}$$

so that multiplying through the numerator and denominator by c, we replace

$$c\sqrt{\frac{\epsilon}{\mu}} = \frac{n}{\mu}$$

and similarly $c\sqrt{\frac{\epsilon'}{\mu'}} = \frac{n'}{\mu'}$,

$$\begin{aligned}
\frac{\mathcal{E}''}{\mathcal{E}} &= \frac{c\sqrt{\frac{\epsilon}{\mu}}\cos i - c\sqrt{\frac{\epsilon'}{\mu'}}\sqrt{1-\sin^2 r}}{c\sqrt{\frac{\epsilon}{\mu}}\cos i + c\sqrt{\frac{\epsilon'}{\mu'}}\sqrt{1-\sin^2 r}} \\
&= \frac{\frac{1}{\mu}n\cos i - \frac{1}{\mu'}n'\sqrt{1-\sin^2 r}}{\frac{1}{\mu}n\cos i + \frac{1}{\mu'}n'\sqrt{1-\sin^2 r}} \\
&= \frac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}
\end{aligned}$$

Now multiply through by μ and use Snell's law to write this in terms of the angle of incidence only as

$$\frac{\mathcal{E}''}{\mathcal{E}} = \frac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}$$

Finally, substituting this result into $\mathcal{E} + \mathcal{E}'' = \mathcal{E}'$, we have

$$\begin{aligned}
\mathcal{E}' &= \mathcal{E} + \mathcal{E}'' \\
\frac{\mathcal{E}'}{\mathcal{E}} &= 1 + \frac{\mathcal{E}''}{\mathcal{E}} \\
&= 1 + \frac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}} \\
&= \frac{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i} + n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}} \\
&= \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}
\end{aligned}$$

so we have the final result for parallel reflection/refraction:

$$\begin{aligned}
\frac{\mathcal{E}'}{\mathcal{E}} &= \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}} \\
\frac{\mathcal{E}''}{\mathcal{E}} &= \frac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}
\end{aligned}$$

Polarization in the plane of incidence

Now let $\boldsymbol{\mathcal{E}}$ lie in the xz-plane. Then, setting

$$\begin{aligned}
\boldsymbol{\mathcal{E}} &= -\mathbf{i}\mathcal{E}\cos i + \mathbf{n}\mathcal{E}\sin i \\
\boldsymbol{\mathcal{E}}' &= -\mathbf{i}\mathcal{E}'\cos r + \mathbf{n}\mathcal{E}'\sin r \\
\boldsymbol{\mathcal{E}}'' &= \mathbf{i}\mathcal{E}''\cos i + \mathbf{n}\mathcal{E}''\sin i
\end{aligned}$$

we substitute into the boundary conditions,

$$\begin{aligned}
\epsilon\left(\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}''\right)\cdot\mathbf{n} &= \epsilon'\boldsymbol{\mathcal{E}}'\cdot\mathbf{n} \\
\left(\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}''\right)\times\mathbf{n} &= \boldsymbol{\mathcal{E}}'\times\mathbf{n} \\
\left(\mathbf{k}\times\boldsymbol{\mathcal{E}} + \mathbf{k}''\times\boldsymbol{\mathcal{E}}''\right)\cdot\mathbf{n} &= \mathbf{k}'\times\boldsymbol{\mathcal{E}}'\cdot\mathbf{n} \\
\frac{1}{\mu}\left(\mathbf{k}\times\boldsymbol{\mathcal{E}} + \mathbf{k}''\times\boldsymbol{\mathcal{E}}''\right)\times\mathbf{n} &= \frac{1}{\mu'}\left(\mathbf{k}'\times\boldsymbol{\mathcal{E}}'\right)\times\mathbf{n}
\end{aligned}$$

For the first,

$$\begin{aligned}
\epsilon\left(\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}''\right)\cdot\mathbf{n} &= \epsilon'\boldsymbol{\mathcal{E}}'\cdot\mathbf{n} \\
\epsilon\left(-\mathbf{i}\mathcal{E}\cos i + \mathbf{n}\mathcal{E}\sin i + \mathbf{i}\mathcal{E}''\cos i + \mathbf{n}\mathcal{E}''\sin i\right)\cdot\mathbf{n} &= \epsilon'\left(-\mathbf{i}\mathcal{E}'\cos r + \mathbf{n}\mathcal{E}'\sin r\right)\cdot\mathbf{n} \\
\epsilon\left(\mathcal{E}\sin i + \mathcal{E}''\sin i\right) &= \epsilon'\mathcal{E}'\sin r
\end{aligned}$$

the second becomes

$$\begin{aligned}
(\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}'') \times \mathbf{n} &= \boldsymbol{\mathcal{E}}' \times \mathbf{n} \\
(-\mathbf{i}\mathcal{E}\cos i + \mathbf{n}\mathcal{E}\sin i + \mathbf{i}\mathcal{E}''\cos i + \mathbf{n}\mathcal{E}''\sin i) \times \mathbf{n} &= (-\mathbf{i}\mathcal{E}'\cos r + \mathbf{n}\mathcal{E}'\sin r) \times \mathbf{n} \\
\mathbf{j}(\mathcal{E}\cos i - \mathbf{i}\mathcal{E}''\cos i) &= \mathbf{j}\mathcal{E}'\cos r \\
\mathcal{E}\cos i - \mathcal{E}''\cos i &= \mathcal{E}'\cos r
\end{aligned}$$

the third is now

$$\begin{aligned}
(\mathbf{k} \times \boldsymbol{\mathcal{E}} + \mathbf{k}'' \times \boldsymbol{\mathcal{E}}'') \cdot \mathbf{n} &= \mathbf{k}' \times \boldsymbol{\mathcal{E}}' \cdot \mathbf{n} \\
(-\mathbf{j}k\mathcal{E} - \mathbf{j}k''\mathcal{E}'') \cdot \mathbf{n} &= -\mathbf{j}k'\mathcal{E}' \cdot \mathbf{n} \\
0 &= 0
\end{aligned}$$

and, finally, the fourth gives

$$\begin{aligned}
\frac{1}{\mu}(\mathbf{k} \times \boldsymbol{\mathcal{E}} + \mathbf{k}'' \times \boldsymbol{\mathcal{E}}'') \times \mathbf{n} &= \frac{1}{\mu'}(\mathbf{k}' \times \boldsymbol{\mathcal{E}}') \times \mathbf{n} \\
\frac{1}{\mu}(-\mathbf{j}k\mathcal{E} - \mathbf{j}k''\mathcal{E}'') \times \mathbf{n} &= \frac{1}{\mu'}(-\mathbf{j}k'\mathcal{E}') \times \mathbf{n} \\
-\frac{1}{\mu}\omega\sqrt{\mu\epsilon}(\mathcal{E} + \mathcal{E}'')\,\mathbf{i} &= -\frac{1}{\mu'}\omega\sqrt{\mu'\epsilon'}\mathcal{E}'\mathbf{i} \\
\sqrt{\frac{\epsilon}{\mu}}(\mathcal{E} + \mathcal{E}'') &= \sqrt{\frac{\epsilon'}{\mu'}}\mathcal{E}'
\end{aligned}$$

Use Snell's law on the first of these four,

$$\begin{aligned}
\epsilon(\mathcal{E}\sin i + \mathcal{E}''\sin i) &= \epsilon'\mathcal{E}'\sin r \\
\frac{\epsilon}{k}(\mathcal{E} + \mathcal{E}'') &= \frac{\epsilon'}{k'}\mathcal{E}'
\end{aligned}$$

$$\begin{aligned}
\frac{\epsilon}{\omega\sqrt{\mu\epsilon}}(\mathcal{E} + \mathcal{E}'') &= \frac{\epsilon'}{\omega\sqrt{\mu'\epsilon'}}\mathcal{E}' \\
\sqrt{\frac{\epsilon}{\mu}}(\mathcal{E} + \mathcal{E}'') &= \sqrt{\frac{\epsilon'}{\mu'}}\mathcal{E}'
\end{aligned}$$

showing that it reproduces the fourth. Therefore, we have just two equations,

$$\begin{aligned}
\mathcal{E}\cos i - \mathcal{E}''\cos i &= \mathcal{E}'\cos r \\
\sqrt{\frac{\epsilon}{\mu}}(\mathcal{E} + \mathcal{E}'') &= \sqrt{\frac{\epsilon'}{\mu'}}\mathcal{E}'
\end{aligned}$$

Substituting the second into the first,

$$\begin{aligned}
\mathcal{E}\cos i - \mathcal{E}''\cos i &= \sqrt{\frac{\mu'}{\epsilon'}}\sqrt{\frac{\epsilon}{\mu}}(\mathcal{E} + \mathcal{E}'')\cos r \\
\mathcal{E}\cos i - \mathcal{E}''\cos i &= \sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}\mathcal{E}\cos r + \sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}\mathcal{E}''\cos r \\
\mathcal{E}''\cos i + \sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}\mathcal{E}''\cos r &= \mathcal{E}\cos i - \sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}\mathcal{E}\cos r
\end{aligned}$$

so that

$$\begin{aligned}
\frac{\mathcal{E}''}{\mathcal{E}} &= \frac{\cos i - \sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}\cos r}{\cos i + \sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}\cos r} \\
&= \frac{\cos i - \frac{n\mu'}{n'\mu}\cos r}{\cos i + \frac{n\mu'}{n'\mu}\cos r} \\
&= \frac{n'^2\cos i - \frac{\mu'}{\mu}nn'\sqrt{1-\sin^2 r}}{n'^2\cos i + \frac{\mu'}{\mu}nn'\sqrt{1-\sin^2 r}} \\
&= \frac{n'^2\cos i - \frac{\mu'}{\mu}n\sqrt{n'^2 - n'^2\sin^2 r}}{n'^2\cos i + \frac{\mu'}{\mu}n\sqrt{n'^2 - n'^2\sin^2 r}} \\
&= \frac{n'^2\cos i - \frac{\mu'}{\mu}n\sqrt{n'^2 - n^2\sin^2 i}}{n'^2\cos i + \frac{\mu'}{\mu}n\sqrt{n'^2 - n^2\sin^2 i}}
\end{aligned}$$

Notice how the last few steps allow us to express $\cos r$ in terms of the incident angle, i.

Finally, substitute rewrite the second boundary condition as

$$\begin{aligned}
\sqrt{\frac{\epsilon}{\mu}}(\mathcal{E}+\mathcal{E}'') &= \sqrt{\frac{\epsilon'}{\mu'}}\mathcal{E}' \\
\sqrt{\frac{\epsilon}{\mu}}\left(1+\frac{\mathcal{E}''}{\mathcal{E}}\right) &= \sqrt{\frac{\epsilon'}{\mu'}}\frac{\mathcal{E}'}{\mathcal{E}} \\
\frac{\mathcal{E}'}{\mathcal{E}} &= \sqrt{\frac{\mu'}{\epsilon'}}\sqrt{\frac{\epsilon}{\mu}}\left(1+\frac{\mathcal{E}''}{\mathcal{E}}\right) \\
&= \frac{\mu'}{\mu}\sqrt{\frac{\mu\epsilon}{\mu'\epsilon'}}\left(1+\frac{\mathcal{E}''}{\mathcal{E}}\right) \\
&= \frac{n\mu'}{n'\mu}\left(1+\frac{\mathcal{E}''}{\mathcal{E}}\right)
\end{aligned}$$

and substitute the first solution,

$$\begin{aligned}
\frac{\mathcal{E}'}{\mathcal{E}} &= \frac{n\mu'}{n'\mu}\left(1+\frac{\mathcal{E}''}{\mathcal{E}}\right) \\
&= \frac{n\mu'}{n'\mu}\left(1+\frac{n'^2\cos i - \frac{\mu'}{\mu}n\sqrt{n'^2 - n^2\sin^2 i}}{n'^2\cos i + \frac{\mu'}{\mu}n\sqrt{n'^2 - n^2\sin^2 i}}\right) \\
&= \frac{n\mu'}{n'\mu}\left(\frac{n'^2\cos i + \frac{\mu'}{\mu}n\sqrt{n'^2 - n^2\sin^2 i} + n'^2\cos i - \frac{\mu'}{\mu}n\sqrt{n'^2 - n^2\sin^2 i}}{n'^2\cos i + \frac{\mu'}{\mu}n\sqrt{n'^2 - n^2\sin^2 i}}\right) \\
&= \frac{n\mu'}{n'\mu}\left(\frac{2n'^2\cos i}{n'^2\cos i + \frac{\mu'}{\mu}n\sqrt{n'^2 - n^2\sin^2 i}}\right) \\
&= \frac{\mu'}{\mu}\left(\frac{2nn'\cos i}{n'^2\cos i + \frac{\mu'}{\mu}n\sqrt{n'^2 - n^2\sin^2 i}}\right) \\
&= \frac{2nn'\cos i}{\frac{\mu}{\mu'}n'^2\cos i + n\sqrt{n'^2 - n^2\sin^2 i}}
\end{aligned}$$

Therefore, the electric field amplitudes for incidence with polarization in the plane of the scattering are given by

$$\begin{aligned}\frac{\mathcal{E}'}{\mathcal{E}} &= \frac{2nn'\cos i}{\frac{\mu}{\mu'}n'^2\cos i + n\sqrt{n'^2 - n^2\sin^2 i}}\\ \frac{\mathcal{E}''}{\mathcal{E}} &= \frac{n'^2\cos i - \frac{\mu'}{\mu}n\sqrt{n'^2 - n^2\sin^2 i}}{n'^2\cos i + \frac{\mu'}{\mu}n\sqrt{n'^2 - n^2\sin^2 i}}\end{aligned}$$

Special cases

There are two special cases. For simplicity, we set $\mu = \mu'$.

Total internal reflection

For polarization perpendicular to the plane of incidence, the direction of the refracted wave is given by Snell's law,

$$n\sin i = n'\sin r$$

and r becomes $\frac{\pi}{2}$ if $n > n'$ and

$$\sin i = \frac{n'}{n}$$

This means that the refracted wave moves only in the x-direction, and not into the second medium at all. We have total internal reflection, and the wave stays in the first medium.

Total absorption

For polarization parallel to the plane of incidence, it is possible for the reflected wave to vanish. With $\mu = \mu'$, the conditon for this is

$$0 = \frac{\mathcal{E}''}{\mathcal{E}}$$

$$\begin{aligned} &= \frac{n'^2\cos i - n\sqrt{n'^2 - n^2\sin^2 i}}{n'^2\cos i + n\sqrt{n'^2 - n^2\sin^2 i}}\\ n'^2\cos i &= n\sqrt{n'^2 - n^2\sin^2 i}\\ n'^4\cos^2 i &= n^2n'^2 - n^4\sin^2 i\\ n'^4 - n'^4\sin^2 i &= n^2n'^2 - n^4\sin^2 i\\ n'^4 - n^2n'^2 &= \left(n'^4 - n^4\right)\sin^2 i\\ \left(n'^2 - n^2\right)n'^2 &= \left(n'^2 - n^2\right)\left(n'^2 + n^2\right)\sin^2 i\\ \frac{n'}{\sqrt{n'^2 + n^2}} &= \sin i\\ \cos i &= \sqrt{1 - \sin^2 i}\\ &= \sqrt{1 - \frac{n'^2}{n'^2 + n^2}}\\ &= \frac{n}{\sqrt{n'^2 + n^2}}\end{aligned}$$

and therefore,

$$\tan i = \frac{n'}{n}$$

This angle, $i_B = \tan^{-1}\frac{n'}{n}$, is called Brewster's angle. When light with a mixture of polarizations strikes a surface at or near Brewster's angle, only the polarization perpendicular to the plane of incidence is reflected.

Wave Equation in Free Space:

Start with Maxwell's equations in derivative form for empty space.

$$\nabla \cdot \mathbf{E} = 0 \quad \text{(Gauss)}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{(no name)}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{(Faraday)}$$

$$\nabla \times \mathbf{B} = \mu_0\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \text{(Ampère)}$$

These equations are first order, which usually means the mathematics should be easy (good!), but they're also coupled, which means it might be difficult (rats!). Let's separate them using this little trick. Take the curl of both sides of Faraday's and Ampère's laws. The left side of each equation is the curl of the curl, for which there is a special identity. The right side of each equation, on the other hand, is the curl of a time derivative. We'll switch it around into a time derivative of the curl.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t}\right)$$

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B})$$

$$\nabla \times \mathbf{B} = \mu_0\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla \times \left(\mu_0\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

$$\nabla(\nabla \cdot \mathbf{B}) - \nabla^2\mathbf{B} = \mu_0\varepsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E})$$

Now if you look carefully, you'll see that one term in each equation equals zero and the other can be replaced with a time derivative.

$$0 - \nabla^2\mathbf{E} = -\frac{\partial}{\partial t}\left(\mu_0\varepsilon_0\frac{\partial \mathbf{E}}{\partial t}\right)$$

$$0 - \nabla^2\mathbf{B} = \mu_0\varepsilon_0\frac{\partial}{\partial t}\left(-\frac{\partial \mathbf{B}}{\partial t}\right)$$

Let's clean it up a bit and see what we get.

$$\nabla^2\mathbf{E} = \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\mathbf{E}$$

$$\nabla^2\mathbf{B} = \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\mathbf{B}$$

These equations are now decoupled (**E** and **B** have their own private equations), which certainly simplifies things, but in the process we've changed them from first to second order (notice all the squares). I know I said earlier that lower order implies easier to work with, but these second order equations aren't as difficult as they look. Raising the order has not made things more complicated; it's made things more interesting.

Here's one set of possible solutions.

$$\mathbf{E}(x,t) = E_0 \sin\left[2\pi(ft - x/\lambda + \varphi)\right]\hat{\mathbf{j}}$$

$$\mathbf{B}(x,t) = B_0 \sin\left[2\pi(ft - x/\lambda + \varphi)\right]\hat{\mathbf{k}}$$

This particular example is one dimensional, but there are two dimensional solutions as well — many of them. The interesting ones have electric and magnetic fields that change in time. These changes then propagate away at a finite speed. Such a solution is an *electromagnetic wave*.

Let's examine our possible solution in more detail. Find the second space and time derivatives of the electric field…

$$\nabla^2\mathbf{E} = -\frac{4\pi^2}{\lambda^2} E_0 \sin\left[2\pi\left(ft - \frac{x}{\lambda} + \varphi\right)\right] \hat{\mathbf{j}}$$

$$\frac{\partial^2}{\partial t^2}\mathbf{E} = -4\pi^2 f^2 E_0 \sin\left[2\pi\left(ft - \frac{x}{\lambda} + \varphi\right)\right] \hat{\mathbf{j}}$$

and substitute them back into the second order partial differential equation.

$$\nabla^2\mathbf{E} = \mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2} \mathbf{E}$$

Work on the left side first.

$$\nabla^2\mathbf{E} = -\frac{4\pi^2}{\lambda^2} E_0 \sin\left[2\pi\left(ft - \frac{x}{\lambda} + \varphi\right)\right] \hat{\mathbf{j}}$$

Work on the right side second.

$$\mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2} \mathbf{E} = \mu_0\varepsilon_0\{-4\pi^2 f^2 E_0 \sin[2\pi(ft - \frac{x}{\lambda} + \varphi)]\} \hat{\mathbf{j}}$$

All kinds of stuff cancels.

$$\frac{1}{\lambda^2} = \mu_0\varepsilon_0 f^2$$

Rearrange a bit.

$$f^2\lambda^2 = 1 / \mu_0\varepsilon_0$$

I see a wave speed in there ($f\lambda$). We'll use c for this one since it's the first letter in the Latin word for swiftness — *celeritas*.

$c = 1 / \sqrt{\mu_0\varepsilon_0}$

Very interesting.

Given Maxwell's four equations (which are based on observation) we have shown that electromagnetic waves must exist as a consequence. They can have any amplitude E_0 (with B_0 depending on E_0 as will be shown later), any wavelength λ, and be retarded or advanced by any phase φ, but they can only travel through empty space at one wave speed c.

$c = 1 / \sqrt{\mu_0\varepsilon_0}$

c = 299,792,458 m/s

In the words of Maxwell…

This velocity is so nearly that of light, that it seems we have strong reasons to conclude that light itself (including radiant heat, and other radiations if any) is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws.

❖ Poynting Theorem and Poynting Vector:

Poynting's theorem is a statement of conservation of energy applied to electromagnetic fields. It helps to interpret the flow of energy with the motion of electromagnetic waves in space. Now,

$$\vec{\nabla} \cdot \left(\vec{E} \times \vec{H}\right) = \vec{H} \cdot \left(\vec{\nabla} \times \vec{E}\right) - \vec{E} \cdot \left(\vec{\nabla} \times \vec{H}\right) \tag{1}$$

Equation (1) is derived using the vector identity $\vec{\nabla}.(\vec{A}_1 \times \vec{A}_2) = \vec{A}_2.(\vec{\nabla} \times \vec{A}_1) - \vec{A}_1.(\vec{\nabla} \times \vec{A}_2)$; Identifying $\vec{A}_1$ with the electric field $\vec{E}$ and $\vec{A}_2$ with the magnetic intensity $\vec{H}$.

Now from Maxwell's 3rd and 4th equation we have,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{2a}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{2b}$$

Using Equations (2a) and (2b) in Equation (1) we have,

$$\vec{\nabla} \cdot \left(\vec{E} \times \vec{H}\right) = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \vec{J} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \tag{3}$$

For any linear medium $\vec{D} = \epsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$. So we can rewrite Eq. (3) as

$$\vec{\nabla} \cdot \left(\vec{E} \times \vec{H}\right) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\vec{E} \cdot \vec{D} + \frac{1}{2}\vec{B} \cdot \vec{H}\right) - \vec{E} \cdot \vec{J}. \tag{4}$$

Integrating this equation over a fixed volume V bounded by a closed surface S and applying Gauss's divergence theorem we get,

$$\oint_S \left(\vec{E}\times\vec{H}\right)\cdot d\vec{S} = -\frac{d}{dt}\int_V \frac{1}{2}\left(\vec{E}\cdot\vec{D}+\vec{B}\cdot\vec{H}\right)\,dV - \int_V \vec{E}\cdot\vec{J}\,dV$$

$$\boxed{\text{or}\quad -\frac{d}{dt}\int_V \frac{1}{2}\left(\vec{E}\cdot\vec{D}+\vec{B}\cdot\vec{H}\right)\,dV = \int_V \vec{E}\cdot\vec{J}\,dV + \oint_S \left(\vec{E}\times\vec{H}\right)\cdot d\vec{S}} \qquad (5)$$

To understand the physical significance of Eq. (5) let us interpret each term in it.

Suppose we have some charge and current configuration which at time **t** produces the $\vec{E}$ and $\vec{B}$ fields.

The rate of work done by the electromagnetic forces on an element of charge $dq=\rho dV$ is given by

$$\frac{dW}{dt} = dq\left(\vec{E}+\vec{v}\times\vec{B}\right)\cdot\vec{v}$$

or,
$$\frac{dW}{dt} = dq\vec{E}\cdot\vec{v} = \vec{E}\cdot\vec{v}(\rho dV) = \vec{E}\cdot\vec{J}\,dV,$$

Where $\vec{J}=\rho\vec{v}$ and $\vec{v}$ is the velocity of the charge element.

Thus, the term

$$\boxed{\int_V \vec{E}\cdot\vec{J}\,dV}$$

in Eq. (5) represents the rate of doing work on the charges in volume ***V*** by the electromagnetic field.

We know that $\frac{1}{2}\vec{E}.\vec{D}$ is the electrostatic energy density and $\frac{1}{2}\vec{B}.\vec{H}$ is the magnetostatic energy density. Hence,

$$\tfrac{1}{2}\left(\vec{E}\cdot\vec{D}+\vec{B}\cdot\vec{H}\right)$$

Thus, the term

$$\boxed{-\frac{d}{dt}\int_V \frac{1}{2}\left(\vec{E}\cdot\vec{D}+\vec{B}\cdot\vec{H}\right)\,dV}$$

in Equation (5) represents the rate at which the total electromagnetic energy in volume Vis decreasing. Physical meaning of the term

$$\oint_S \left(\vec{E}\times\vec{H}\right)\cdot d\vec{S}$$

Now follows from the principle of conservation of energy.
The rate of decrease of electromagnetic field energy within a certain volume is equal to the rate of work done by the field on the charges inside the given volume plus the rate of outflow of electromagnetic energy through the surface bounding the volume. This statement is known as Poynting' theorem.
So the term

$$\oint_S \left(\vec{E} \times \vec{H}\right) \cdot d\vec{S}$$

represents the rate of flow of electromagnetic energy outward through the surface s. The vector $\vec{s} = \vec{E} \times \vec{H}$ represents the amount of electromagnetic energy flowing out normally, through unit area per unit time. This vector $\vec{s}$ is known as Poynting's vector.

Differential form of Poynting's theorem:

The work done by the electromagnetic field on the charges increases their mechanical energy. If we denote the mechanical energy density by u_M then we can write

$\partial u_M / \partial t$ = rate of work done on charges per unit volume = $\vec{E} \cdot \vec{J}$

Now denoting the electromagnetic energy density $\frac{1}{2}(\vec{E}.\vec{D}+\vec{B}.\vec{H})$ by u_{em} we can write Eq. (4) in the following form:

$$\boxed{\frac{\partial}{\partial t}(u_M + u_{em}) + \vec{\nabla} \cdot \vec{s} = 0.}$$

This is the differential form of Poynting's theorem.

This has the same form as the equation of continuity expressing the conservation of charge, with the total energy density taking the place of charge density ρ and $\vec{s}$ taking the place of current density $\vec{J}$.

Therefore, from analogy with $\vec{J}$ the Poynting's vector can be interpreted as the energy flowing through unit area per unit time.