K.T.S.P. Mandal's

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Third Year B.Science

V– Sem (2019 CBCS Pattern)
As per the new syllabus

Subject- Classical Mechanics

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Topic : Lagarangian Equation from D' Alembert's Principle

D'Alembert's principle of virtual work

If virtual work done by the constraint forces is ($\vec{f}_c \cdot \eth\vec{r} = 0$) (from eq.-1),

$$(\vec{F}_e - m\ddot{\vec{r}}) \cdot \eth\vec{r} = 0$$ → **D'Alembert's principle of Virtual work**

Now, for a general system of N particles having virtual displacements, $\eth\vec{r}_1, \eth\vec{r}_2, \ldots, \eth\vec{r}_N$,

$$\sum^{N} (\vec{F}_{ie} - m_i\ddot{\vec{r}}_i) \cdot \eth\vec{r}_i = 0$$

$\vec{F}_{ie}$ →Applied force on i_{th} particle

Does not necessarily means that individual terms of the summation are zero as $\vec{r}_i$ are not independent, they are connected by constrain relation

Lagrange's equation from D'Alembert's principle

- D'Alembert's principle,

$$\sum_{i=1}^{N} (\vec{F}_{ie} - m_i \ddot{\vec{r}}_i) \cdot \delta \vec{r}_i = 0$$

Constraint forces are out of the game!

Now, no need of additional subscript, we shall simply write $\vec{F}_i$ instead of $\vec{F}_{ie}$

But How to express this relation so that individual terms in the summation are zero?

Switch to generalized coordinate system as they are independent!

Let's take the 1st term

$$\sum_i \vec{F}_i \cdot \delta \vec{r}_i = \sum_i \vec{F}_i \cdot \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = \sum_{j=1}^{n} \left(\sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) \delta q_j = \sum_{j=1}^{n} Q_j \delta q_j$$

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

→ **Generalized force**

- Dimensions of Q_j **is not** always of force!
- Dimensions of $Q_j \delta q_j$ **is** always of work!

Lagrange's equation from D'Alembert's principle

- 2nd Term: $$\sum_i m_i \ddot{r}_i \cdot \delta r_i = \sum_i m_i \ddot{r}_i \cdot \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j}\delta q_j = \sum_{i,j} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j$$

- **Bit of rearrangement in derivatives**

$$\ddot{r}_i \cdot \frac{\partial r_i}{\partial q_j} = \frac{d}{dt}\left(\dot{r}_i \cdot \frac{\partial r_i}{\partial q_j}\right) - \dot{r}_i \cdot \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right)$$

Time and coordinate derivative can be interchanged!

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \left(\frac{\partial \dot{\vec{r}}_i}{\partial q_j}\right)$$

$$= \frac{d}{dt}\left(\dot{r}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}\right) - \dot{r}_i \cdot \left(\frac{\partial \dot{\vec{r}}_i}{\partial q_j}\right)$$

dot cancellation!

$$\frac{\partial \dot{r}_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}$$

$$= \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}\dot{r}_i^{\,2}\right)\right\} - \frac{\partial}{\partial q_j}\left(\frac{1}{2}\dot{r}_i^{\,2}\right)$$

Lagrange's equation from D'Alembert's principle

- Thus 2nd term becomes

$$\sum_{i=1}^{N} m_i \mathbf{r}_i \cdot \delta \mathbf{r}_i = \sum_{i,j} \left[\frac{d}{dt}\left\{ \frac{d}{dq_j}\left(\frac{1}{2} \dot{r}^{2} \right) \right\} - \frac{\partial}{\partial q}\left(\frac{1}{2} r_i \right) \right] \delta q$$

$$= \sum_{j} \left[\frac{d}{dt}\left\{ \frac{\partial}{\partial q_j}\left(\sum_{i} \frac{1}{2} \quad \cdot^{2} \right) \right\} - \frac{\partial}{\partial q_j}\left(\sum_{i} \frac{1}{2} \quad ^{2} \right) \right]$$

$$= \sum_{j} \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial q_i} \right) - \frac{\partial T}{\partial q} \right\} \delta q_j$$

The 1st term

$$\sum_{i} \mathbf{F}_i \cdot \delta \mathbf{r}_i = \sum_{j=1}^{n} Q_j \, \delta q_j$$

Lagrange's equation from D'Alembert's principle

- D'Alembert's principle in generalized coordinates becomes

$$\sum_j \left\{\frac{d}{dt}\left(\frac{\partial T}{\partial q}\right) - \frac{\partial T}{\partial q}\right\}\delta q_j = \sum_j Q_j \delta q_j$$

$$\sum_j \left[\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial q}\right) - \frac{\partial T}{\partial q}\right\} - Q_j\right]\delta q_j = 0$$

Well, we are very close to Lagrange's equation!

- Since generalized coordinates q_j are all independent each term in the summation is zero

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q}\right) - \frac{\partial T}{\partial q} = Q_j$$

$$-\left(\frac{\partial V_i}{\partial x_i}\hat{i} + \frac{\partial V_i}{\partial y_i}\hat{j} + \frac{\partial V_i}{\partial z_i}\hat{k}\right)\cdot\left(\frac{\partial x_i}{\partial q}\hat{i} + \frac{\partial y_i}{\partial q}\hat{j} + \frac{\partial z_i}{\partial q}\hat{k}\right)$$

$$= -\left(\frac{\partial V_i}{\partial x_i}\frac{\partial x_i}{\partial q_j} + \frac{\partial V_i}{\partial y_i}\frac{\partial y_i}{\partial q_j} + \frac{\partial V_i}{\partial z_i}\frac{}{}\right)$$

- If all the forces are conservative, then $\vec{F}_i = -\vec{\nabla}V_i$

$$Q_j = \sum_i \left(-\vec{\nabla}V_i\right)\cdot\frac{\partial \vec{r}_i}{\partial q} = -\sum_i \frac{\partial V_i}{\partial q} = -\frac{\partial}{\partial q_i}\sum_i V_i = -\frac{\partial V}{\partial q}$$

Total potential

$$V = \sum_i V_i$$

Lagrange's equation from D'Alembert's principle

Hence,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j = -\frac{\partial V}{\partial q_j}$$

- Assume that V **does not depend on** $\dot{q}_j$, then $\frac{\partial V}{\partial \dot{q}_j} = 0$

$$\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_j}(T - V)\right\} - \frac{\partial (T - V)}{\partial q_j} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

Where,

$$L(q_j, \dot{q}_j, t) = T(q_j, \dot{q}_j, t) - V(q_j, t)$$

We have reached to Lagrange's equation from D'Alembert's principle.

Review of the steps we followed

- Started from Newton's law

$$m\ddot{\vec{r}} = \vec{F}_e + \vec{f}_c$$

- Taken dot product with virtual displacement to kick out constrain force from the game by using $\vec{f}_c \cdot \delta\vec{r} = 0$; Arrive at D'Alembert's principle $\left(\vec{F}_e - m\ddot{\vec{r}}\right) \cdot \delta\vec{r} = 0$

- Extended D'Alembert's principle for a system of particles;

$$\sum_{i=1}^{N} \left(\vec{F}_{ie} - m_i\ddot{\vec{r}}_i\right) \cdot \delta\vec{r}_i = 0$$

- Converted this expression in generalized coordinate system that *"every"* term of this summation is zero to get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q} = Q_j$$

This is a more general expression!

- Now, with the assumptions: i) Forces are conservative, $\vec{F}_i = -\vec{\nabla} V_i$, hence $Q_j = -\frac{\partial V}{\partial q_j}$ and ii) potential does not depend on $\dot{q}_j$, then $\frac{\partial V}{\partial \dot{q}_j} = 0$

We get back our Lagrange's eqn.,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

Discussion on generalized force

- ❑ A system may experience both conservative, non-conservative forces

  i,e. $\vec{F}_i = \vec{F}_i^{\,c} + \vec{\;}^{\,nc}$

- ❑ Hence generalized force for the system

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q} = \sum_i \left(\vec{F}_i^{\,c} + \vec{F}_i^{\,nc}\right) \cdot \frac{\partial \vec{r}_i}{\partial q} = \sum_i \vec{F}_i^{\,c} \cdot \frac{\partial \vec{r}_i}{\partial q} + \sum_i \vec{F}_i^{\,nc} \cdot \frac{\partial \vec{r}_i}{\partial q}$$

$$Q_j = Q_j^{\,c} + Q_j^{\,nc}$$

$$Q_j^{\,c} = \sum_i \vec{F}_i^{\,c} \cdot \frac{\partial \vec{r}_i}{\partial q}$$

- ❑ Generalized force corresponding to conservative part

$$Q_j^{\,nc} = \sum_i \vec{F}_i^{\,nc} \cdot \frac{\partial \vec{r}_i}{\partial q}$$

- ❑ Generalized force corresponding to non-conservative part

Lagrange's equation with both conservative and non-conservative force

- If system may experience both conservative, non-conservative forces

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_i}\right)-\frac{\partial T}{\partial q}=Q_j{}^{c}+Q_j{}^{nc}$$

- Generalized force corresponding to conservative force can be derived from potential $Q_j{}^{c}=-\frac{\quad}{\partial q_j}$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_i}\right)-\frac{\partial T}{\partial q}=-\frac{\partial V}{\partial q}+Q_j{}^{nc}$$

$$\frac{d}{dt}\left\{\frac{\partial}{\partial q_j}(T-V)\right\}-\frac{\partial (T-V)}{\partial q_j}=Q_j{}^{nc}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i}\right)-\frac{\partial L}{\partial q}=Q_j{}^{nc}$$

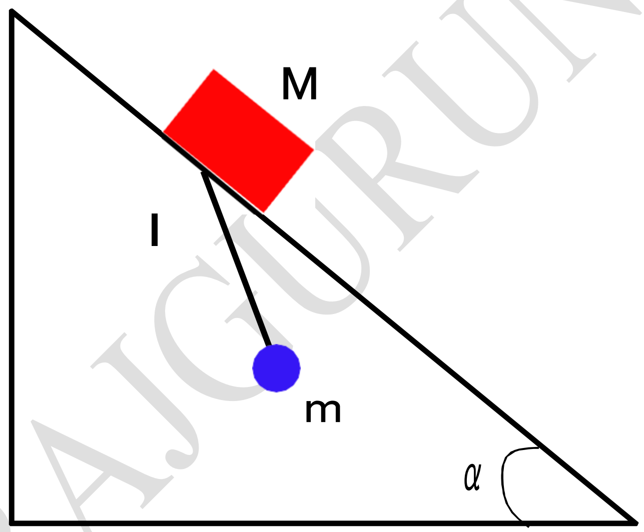
- Assume that V does not depend on q_j, then $\frac{\partial V}{\partial q_j}=0$

$$L=T-V$$

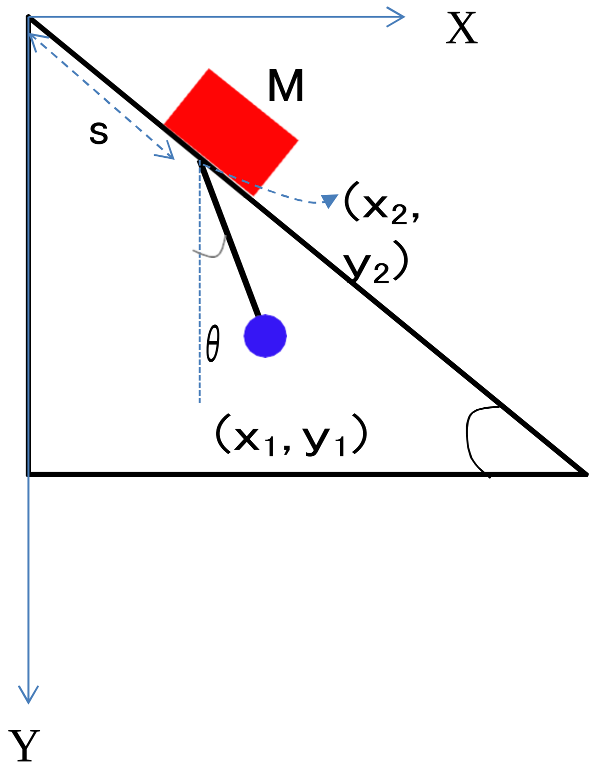
More on Lagrange's equations

Example-5

Example 5: A mass M slides down a frictionless plane inclined at angle α. A pendulum, with length l, and mass m, is attached to M. Find the equations of motion. For small oscillation



Example-5



Four constrains equations

$$z_1 = 0; \; z_2 = 0$$

$$y_2 = x_2 \tan \alpha$$

$$(y_2 - y_1)^2 + (x_2 - x_1)^2 = l^2$$

Step-1: *Find the degrees of freedom and choose suitable generalized coordinates*

Two particles N = 2, no. of constrains (k) = 4

thus degrees of freedom = 3 × 2 − 4 = 2

Hence number of generalized coordinates must be two.

‘s’ and ‘θ’ can serve as generalized coordinates (they are independent nature)

Example-5 continued ….

Step-2: *Find out transformation relations*

$x_2 = s\cos\alpha\,;\ y_2 = s\sin\alpha$

$x_1 = s\cos\alpha + l\sin\theta\ ;\ y_1 = s\sin\alpha + l\cos\theta$

All the constrains relations have been included in the problem through these relationship

Step-3: *Write* T and V *in Cartesian*

$$T = \frac{1}{2}m\left(x_1{}^2 + {}_1{}^2\right) + \frac{1}{2}M\left(x_2{}^2 + {}_2{}^2\right)$$

$V = -mgy_1 -$

Step-4:Convert
T and V in generalized coordinate using transformation

From transformation equation

$x_2 = s\cos\alpha\,;\ y_2 = s\sin\alpha$

$x_1 = s\cos\alpha + l\cos\theta\ \theta;$

$y_1 = s\sin\alpha - l\sin\theta\ \theta$

$$T = \frac{1}{2}m[s^2 + l^2\theta^2 + 2ls\theta\cos(\alpha + \theta)] + \frac{1}{2}$$

$V = -mg(s\sin\alpha + l\cos\theta) - Mgs\sin\alpha$

Example-5 continued ….

Step-5: *Write down Lagrangian*

$$L = T - V$$

$$L = \frac{1}{2} m[s^2 + l^2\theta^2 + 2l\dot{s}\dot{\theta}\cos(\alpha + \theta)] + \frac{1}{2}$$

$$+ mg(s \sin\alpha + l\cos\theta) + Mgs\ \sin\alpha$$

Step-5: *Write down Lagrange's equation for each generalized coordinates*

$$\frac{d}{dt}\left(\frac{\$L}{\$s}\right) - \frac{\$L}{\$s} = 0 \text{ and } \frac{d}{dt}\left(\frac{\$L}{\$\theta}\right) - \frac{\$L}{\$\theta} = 0$$

From 1st eqn

$$\frac{d}{dt}[m\dot{s} + ml\dot{\theta}\cos(\alpha + \theta) + M\dot{s}] - mg\sin\alpha - Mg\sin\alpha = 0$$

$$(m + M)\ddot{s} + ml\ddot{\theta}\ \cos(\alpha + \theta) + ml\dot{\theta}^2\sin(\alpha + \theta) - (m + M)g\sin\alpha = 0$$

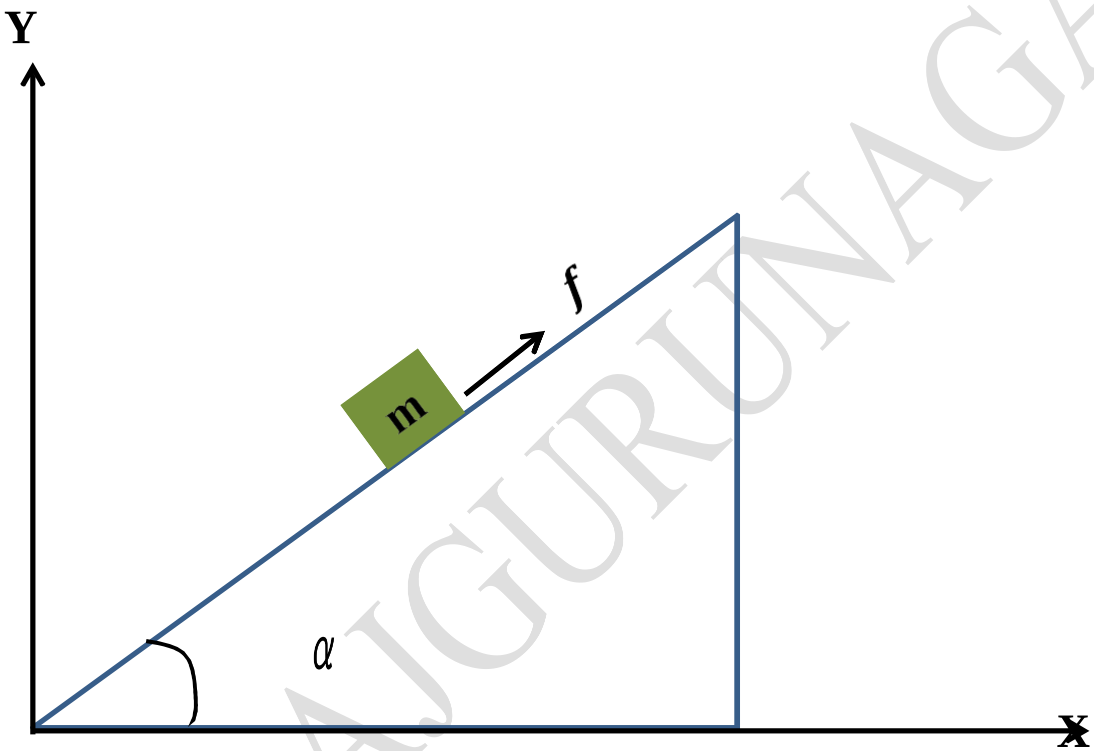
From 2nd eqn

$$\frac{d}{dt}[ml^2\dot{\theta} + ml\dot{s}\ \cos(\alpha + \theta)] + ml\dot{s}\dot{\theta}\sin(\alpha + \theta) + mgl\sin\theta = 0$$

$$ml^2\ddot{\theta} + ml\ddot{s}\cos(\alpha + \theta) + mgl\sin\theta = 0$$

Problems with generalized force

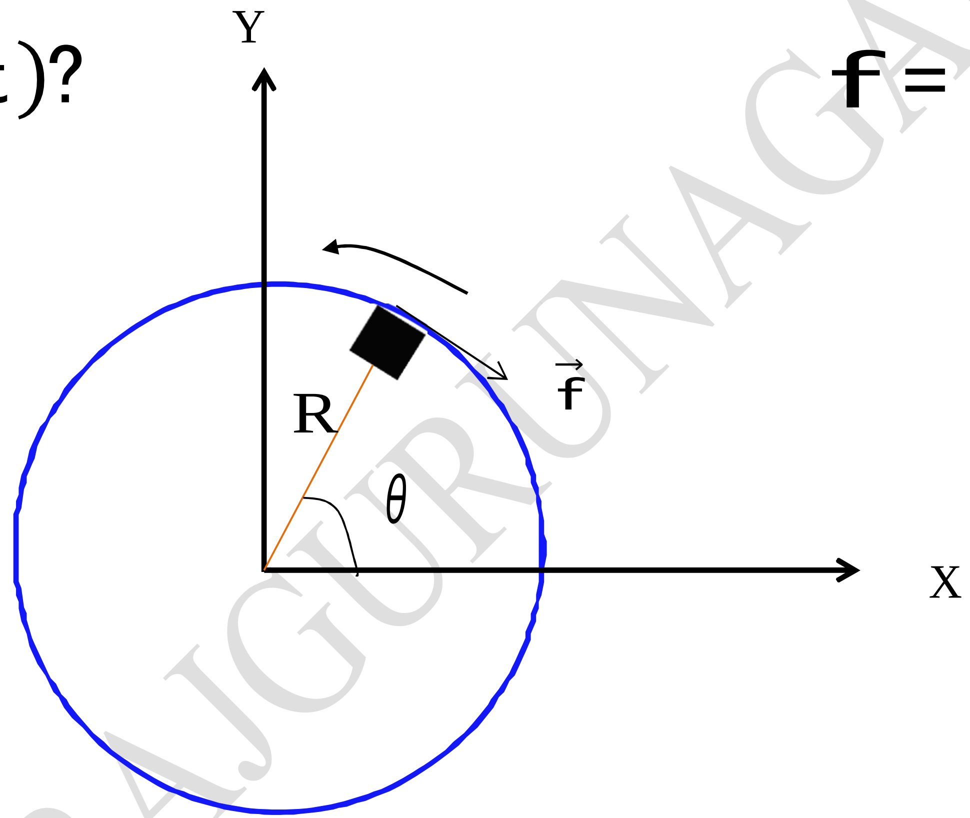
Example-6



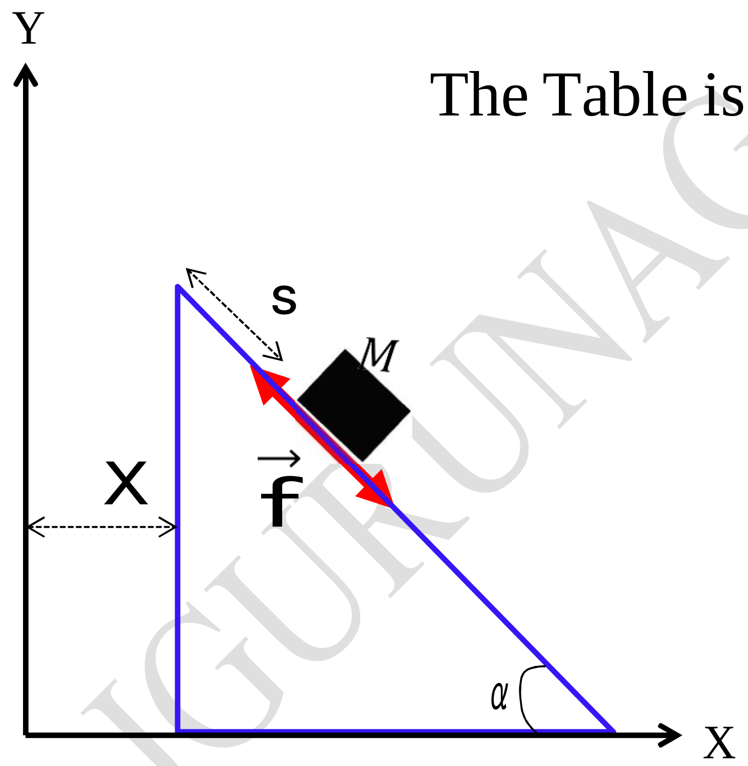
Example-7; Ring & mass on horizontal plane

Find θ (t)?

$\mathbf{f} = \mathbf{C}\theta^2$



Example-8; Wedge & Block under friction, *f*



The Table is friction less!

Generalized coordinate (X, s)