Complex Number

The **real numbers** include all the rational **numbers**, such as the integer −5 and the fraction 4/3, and all the irrational **numbers**, such as √2 (1.41421356..., the square root of 2, an irrational algebraic **number**).

A **rational number:-Number** that can be expressed as the quotient or fraction p/q of two integers, a numerator p and a non-zero denominator q.

Ex:- 5/2, 2/3, 7/5 etc.

Irrational Numbers:- A real number that cannot be written as a simple fraction. Irrational means not Rational. Ex:- √2 , √7 etc.

Applications of Complex number in Physics

- **Electromagnetism and electrical engineering**

  In electrical engineering, the Fourier transform is used to analyze varying voltages and currents.

- **Fluid dynamics :-T**o describe potential flow in two dimensions.

- **Quantum mechanics:-** Schrödinger equation and Heisenberg's matrix mechanics – make use of complex numbers.

Complex Number

A complex number is number consisting of a Real & imaginary part.

It can be written in the form,

$$Z= a+ib$$

Where,

a = Real Part,

b = Imaginary part

The value of $i^2 = -1$

$$i = \sqrt{-1}$$

Notation Used in Complex Number

- A real number *a* can be regarded as a complex number $a + 0i$ whose imaginary part is 0.

- A purely Imaginary number <u>*bi*</u> is a complex number $0 + bi$ whose real part is zero.

- The real part of a complex number *z* is denoted by Re(z); the imaginary part of a complex number *z* is denoted by Im(*z*)

For example, Z1= (2+3i)

Re(Z1)= 2 ; Im (z1)=3

Why Complex Numbers are Introduced?

If we consider Quadratic equation,

$X^2+1=0$ ……………………………………………………………(1)

The roots of quadratic equation is given by,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$D = b^2 - 4ac$

D>0, Two real solutions
D=0, One real solution
D<0, Complex Solutions

D= Discriminant

So roots of quadratic equation (1) are,

$X = \sqrt{-1}$

This root is neither rational nor irrational real number.

As discriminant is negative roots are obtained by introducing new kind of number called Complex Number.

How to get Roots of quadratic Equation

$$y = x^2 - 4x + 5$$

$$\frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)}$$

$$\downarrow$$

$$\frac{4 \pm \sqrt{16 - 20}}{2} \rightarrow \frac{4 \pm \sqrt{-4}}{2}$$

$$\frac{4 \pm 2i}{2} \nearrow \frac{4 + 2i}{2} \rightarrow 2 + i$$

$$\searrow \frac{4 - 2i}{2} \rightarrow 2 - i$$

How to get quadratic Equation from Roots

Write an equation from the Roots

Find the equation of a quadratic function that has the following numbers as roots:

Distribute.

$2 - 3i$ and $2 + 3i$

$$y = (x - (2 - 3i))(x - (2 + 3i))$$

$$y = x^2 - x(2 + 3i) - x(2 - 3i) + (2 - 3i)(2 + 3i)$$

$$y = x^2 - 2x - 3xi - 2x + 3xi + 4 + 6i - 6i - 9i^2$$

$$y = x^2 - 4x + 4 - 9(-1)$$

$$y = x^2 - 4x + 13$$

Complex Conjugates

If Z = x+iy is complex number then its complex conjugate is denoted by,

$$Z^* = x - iy$$

The Modulus OR absolute Value is given by,

$$ZZ^* = |Z| = \sqrt{(x^2+y^2)}$$

Complex Conjugates

Complex conjugates are a pair of complex numbers of the form $a + bi$ and $a - bi$ where a and b are real numbers.

The product of a complex conjugate pair is a positive real number.

$$(a+bi)(a-bi)$$
$$= a^2 - abi + abi - b^2 i^2$$
$$= a^2 - b^2(-1)$$
$$= a^2 + b^2$$

Finding Modulus of Complex Number

Example 8.6.3: Finding the Absolute Value of a Complex Number

Given $z = 3 - 4i$, find $|z|$.

Solution

Using the formula, we have

$$|z| = \sqrt{x^2 + y^2}$$
$$|z| = \sqrt{(3)^2 + (-4)^2}$$
$$|z| = \sqrt{9 + 16}$$
$$|z| = \sqrt{25}$$
$$|z| = 5$$

Find the absolute value of $z = \sqrt{5} - i$.

Solution

Using the formula, we have

$$|z| = \sqrt{x^2 + y^2}$$
$$|z| = \sqrt{(\sqrt{5})^2 + (-1)^2}$$
$$|z| = \sqrt{5+1}$$
$$|z| = \sqrt{6}$$

Equal Complex Numbers

Two **complex numbers** are **equal** if and only if their real parts are **equal** and their imaginary parts are **equal**.

Consider two complex numbers $Z_1 = x_1+iy_1$, $Z_2 = x_2+iy_2$, if $x_1 = x_2$ & $y_1 = y_2$ then $Z_1 = Z_2$

Equal Complex Number

- Consider two complex numbers,

  $Z_1 = 3+2i$, $Z_2 = 3+2i$

If $Re(Z_1) = Re(Z_2)$ & $Im(Z_1) = Im(Z_2)$ then complex numbers are said to be equal.

$Re(z_1) = 3 = Re(z_2)$, Also

$Im(z_1) = 2 = Im(z_2)$

Addition of Complex Number

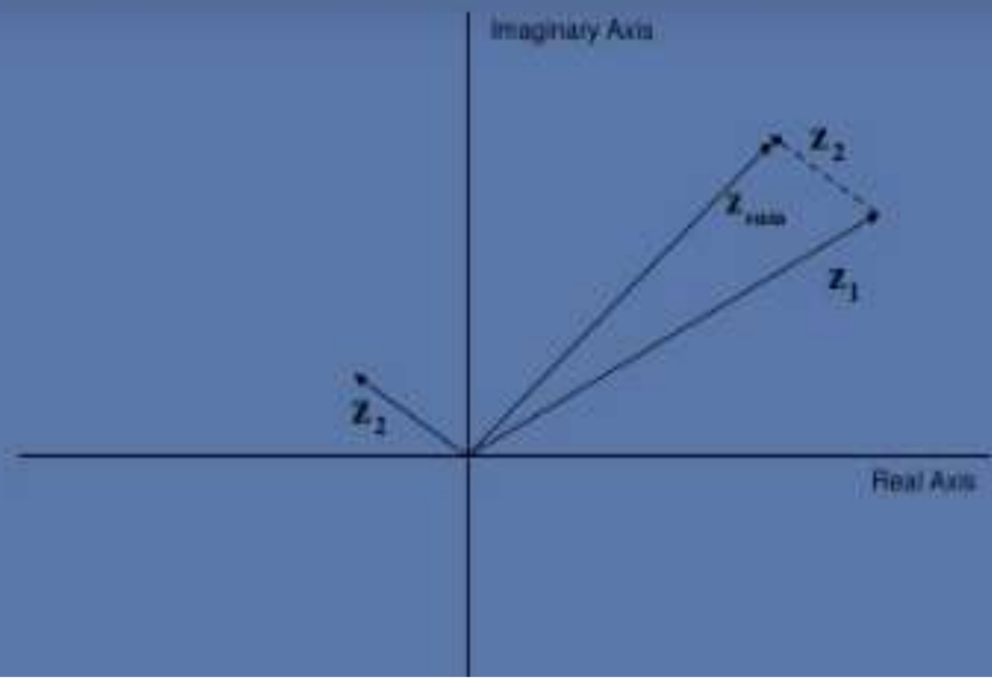
$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

EXAMPLE

$$(2+3i)+(1+5i)$$

$$=(2+1)+(3+5)i$$

$$=3+8i$$

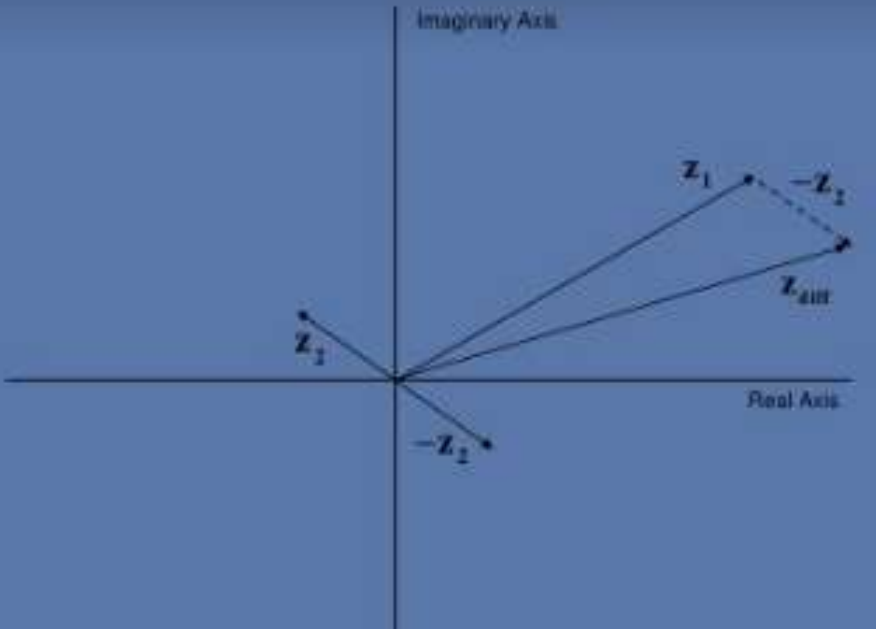


Subtraction of Complex Number

$$(a+bi)-(c+di)=(a-c)+(b-d)i$$

Example

$$(2+3i)-(1+5i)$$
$$=(2-1)+(3-5)i$$
$$=1-2i$$



Multiplication of Complex Number

$$(a+bi)(c+di)=(ac-bd)+(ad+bc)i$$

Example

$$(2+3i)(1+5i)$$

$$=(2-15)+(10+3)i$$

$$=-13+13i$$

Example :- 1

$$(3 + 2i)(1 - 4i) = 3 - 12i + 2i - 8i^2$$

$$= 3 - 10i - 8(\sqrt{-1})^2$$

$$= 3 - 10i - 8(-1)$$

$$= 3 - 10i + 8$$

$$= 11 - 10i$$

Example :- 2

$$(2-3i)(2+3i)$$
$$=4+6i-6i-9i^2$$
$$=4-9(-1)$$
$$=4+9$$
$$=13$$

Example :- 3

$$
\begin{aligned}
(2+5i)(4-3i) &= 8-6i+20i-15i^2 \\
&= 8+14i-15i^2 \\
&= 8+14i-15(-1) \\
&= 8+14i+15 \\
&= 23+14i \checkmark
\end{aligned}
$$

Division of Complex Number

$$\frac{(a+bi)}{(c+di)} = \frac{(a+bi)}{(c+di)} \bullet \frac{(c-di)}{(c-di)}$$

$$= \frac{ac-adi+bci-bdi^2}{c^2+d^2}$$

$$= \frac{ac+bd+(bc-ad)i}{c^2+d^2}$$

EXAMPLE

$$\frac{(6-7i)}{(1-2i)} = \frac{(6-7i)}{(1-2i)} \bullet \frac{(1+2i)}{(1+2i)}$$

$$= \frac{6+12i-7i-14i^2}{1^2+2^2} = \frac{6+14+5i}{1+4}$$

$$= \frac{20+5i}{5} = \frac{20}{5} + \frac{5i}{5} = 4+i$$

Example :-2

$$\frac{2+3i}{4-5i} = \frac{2+3i}{4-5i}\cdot\frac{4+5i}{4+5i} =$$

$$= \frac{8+10i+12i+15i^2}{16+20i-20i-25i^2} =$$

$$= \frac{8+22i+15\cdot(-1)}{16-25\cdot(-1)} =$$

$$= \frac{-7+22i}{49} =$$

$$= -\frac{7}{49}+\frac{22}{49}i$$

Example :-3

$$\left(\frac{7+4i}{-3-i}\right)\left(\frac{-3+i}{-3+i}\right)=\frac{-21+7i-12i+4i^2}{9-3i+3i-i^2}$$

$$=\frac{-21-5i+4i^2}{9-i^2}$$

$$=\frac{-21-5i+4(-1)}{9-(-1)}$$

$$=\frac{-21-5i-4}{10}$$

$$=\frac{-25-5i}{10}$$

$$=\frac{5(-5-i)}{10}$$

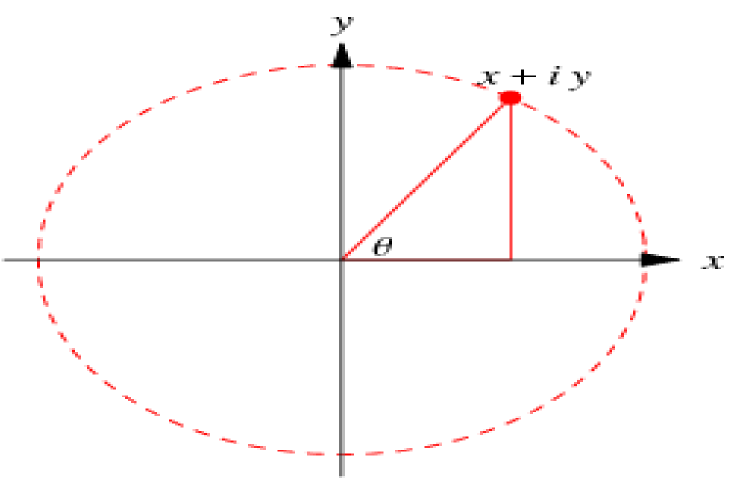
$$\left(\frac{7+4i}{-3-i}\right)\left(\frac{-3+i}{-3+i}\right)=\frac{-5-i}{2}$$ ✓

Argand Diagram

An Argand diagram is a plot of complex number as points.

In the Complex Plane(x-y) using the x-axis as the real axis and y-axis as the imaginary axis.

In the plot above, the dashed circle represents the complex modulus of and the angle represents its complex argument.

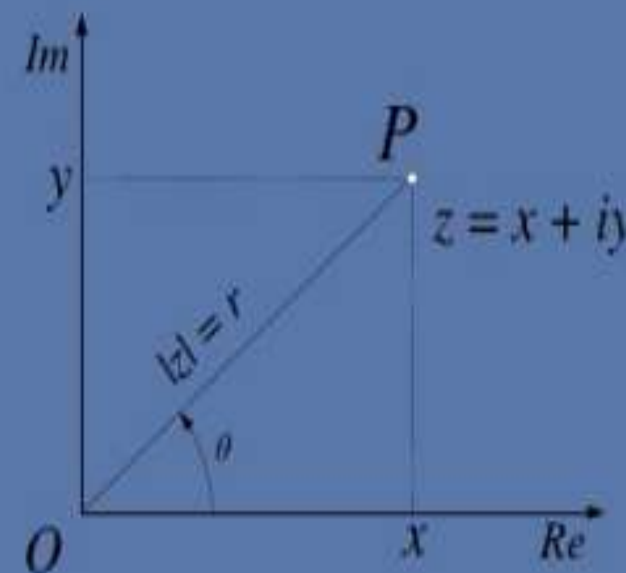


Geometrically, |z| is the distance of the point z from the origin while θ is the directed angle from the positive x-axis to OP in the above figure.

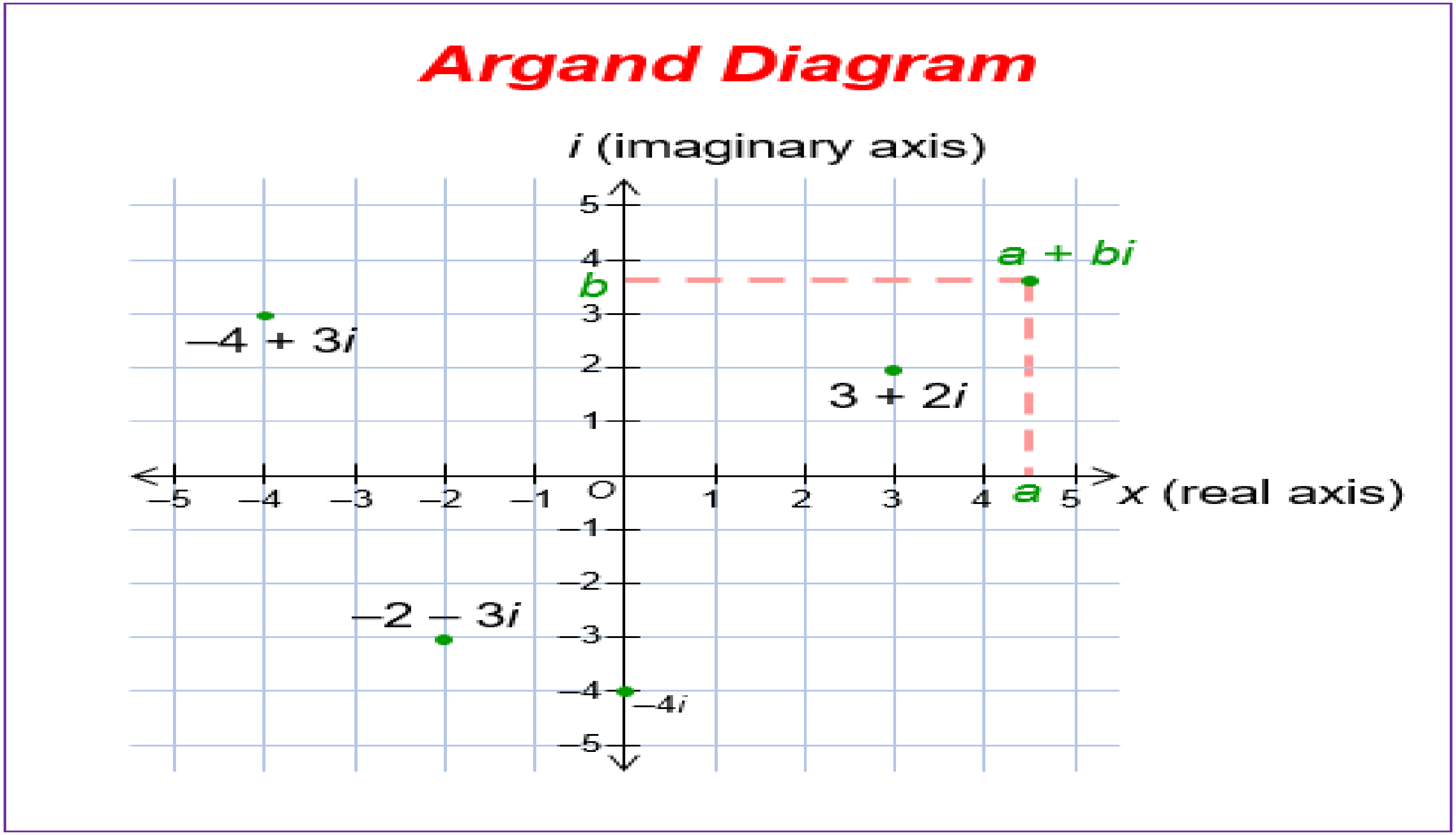
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

θ is called the **argument** of z and is denoted by arg z. Thus,

$$\theta = \arg z = \tan^{-1}\left(\frac{y}{x}\right) \qquad z \neq 0$$



Representing Complex No. on Argand Diagram



Rectangular Form

The **form** $z=a+bi$ is called the **rectangular** coordinate **form of a complex number**.

The horizontal axis is the real axis and the vertical axis is the imaginary axis.

In the case of a **complex number**, r represents the absolute value or modulus and the angle θ is called the argument of the **complex number**.

Complex number in Polar Form

Convert Complex Number from Rectangular Form to Polar (Trigonometric) Form

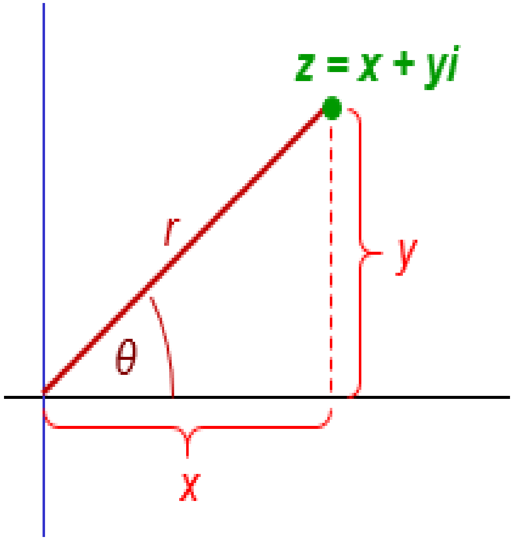
$z = x + yi$ (rectangular form)

$$r = |z| = \sqrt{x^2 + y^2}$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$z = r(\cos\theta + i\sin\theta)$ (polar form)



- $\Theta = \tan^{-1}(y/x)$

If $Z = 0+3i$ then , $x = 0$, $y = 3$

- $\Theta = \tan^{-1}(y/x) = \tan^{-1}(3/0) = \infty$

$\Theta = \pi/2$

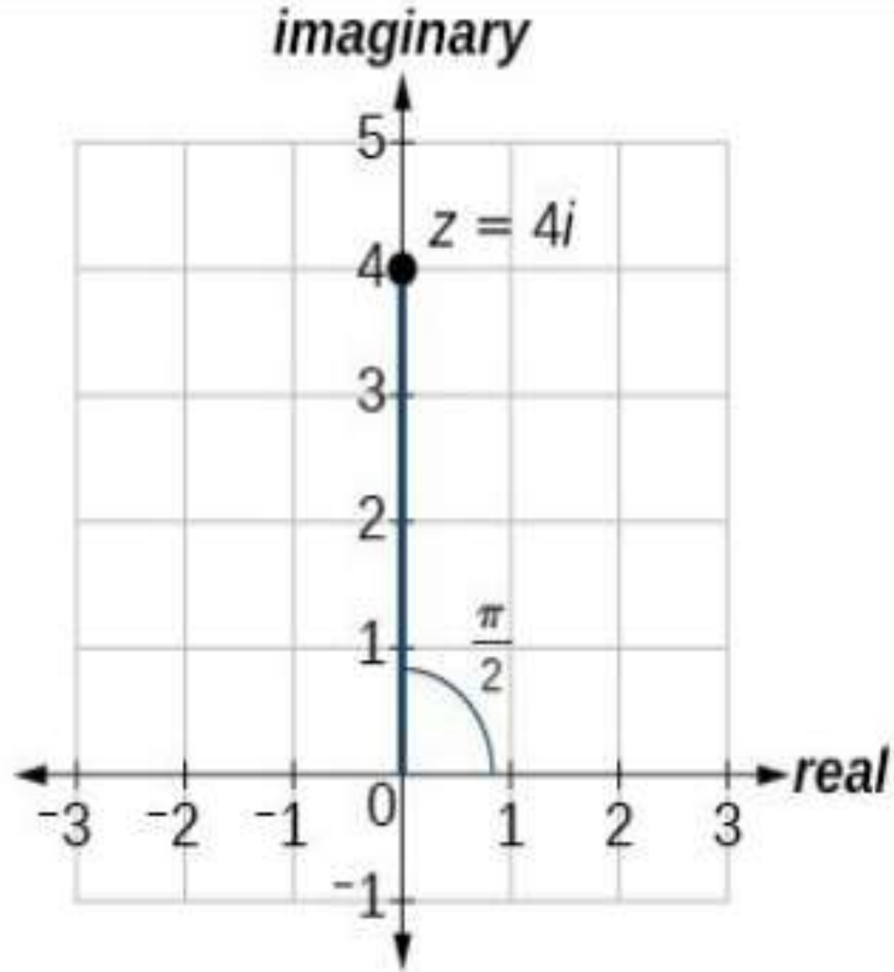
Express the complex number $4i$ using polar coordinates.

Solution

On the complex plane, the number $z = 4i$ is the same as $z = 0 + 4i$. Writing it in polar form, we have to calculate r first.

$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{0^2 + 4^2}$$
$$r = \sqrt{16}$$
$$r = 4$$

Next, we look at x. If $x = r\cos\theta$, and $x = 0$, then $\theta = \frac{\pi}{2}$. In polar coordinates, the complex number $z = 0 + 4i$ can be written as $z = 4\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$ or $4\,cis\left(\frac{\pi}{2}\right)$. See Figure 8.6.7.


imaginary
real
z = 4i
π/2
5
4
3
2
1
0
−1
−3
−2
−1
1
2
3

Find the polar form of $-4+4i$.

SOLUTION

First, find the value of r.

$$r=\sqrt{x^2+y^2}$$
$$r=\sqrt{(-4)^2+(4^2)}$$
$$r=\sqrt{32}$$
$$r=4\sqrt{2}$$

Find the angle θ using the formula:

$$\cos\theta=\frac{x}{r}$$
$$\cos\theta=\frac{-4}{4\sqrt{2}}$$
$$\cos\theta=-\frac{1}{\sqrt{2}}$$
$$\theta=\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)=\frac{3\pi}{4}$$

$Z = 4\sqrt{2}(\cos(3\pi/4)+ i\sin(3\pi/4))$

Express 1 in Polar form

- $Z = x+iy$
- $Z = 1+0i$

Where,

$X = 1$, $y = 0$

So

$\Theta = \tan^{-1}(y/x)$

$\Theta = \tan^{-1}(0/1) = \pi$

Also $r = 1 =$ modulus

$Z = 1(\cos\pi + i\sin\pi)$

Express $-\pi i$ in Polar form

$Z = x+iy$

$Z = -\pi i$

Where,

$X = 0$, $y = -\pi$

So

$\Theta = \tan^{-1}(y/x)$

$\Theta = \tan^{-1}(-\pi/0) = \tan^{-1}(\infty) = \pi/2$

Also $r = =$ modulus $= ZZ^* = |Z| = \sqrt{(x^2+y^2)} = \pi$

$Z = \pi(\cos\pi/2 + i\sin\pi/2)$

Complex number in Exponential Form

Euler Formula :-

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Way to write complex number in exponential form,

$$Z = r\,e^{i\theta}$$

- Express the number $z = 3 + 3j$ in exponential form.
- Step1:-

First find its modulus and argument.

$z = 3 + 3j$

$X = 3, y = 3$

$r = \sqrt{(3^2 + 3^2)}$

$= \sqrt{18}$.

Step2:- Finding argument θ

The complex number lies in the first quadrant of the Argand diagram and so its argument θ is given by $\theta = \tan^{-1}(3/3) = \pi/4$.

Thus $z = 3 + 3j = \sqrt{18}e^{i\pi/4}$

Express πi in Exponential form

- $Z = x+iy$
- $Z = \pi i$

Where,

$X = 0$, $y = \pi$

So

$$\Theta = \tan^{-1}(y/x)$$

$$\Theta = \tan^{-1}(\pi/0) = \tan^{-1}(\infty) = \pi/2$$

Also $r = =$ modulus $= ZZ^* = |Z| = \sqrt{(x^2+y^2)} = \pi$

$$Z = r\, e^{i\theta} = \pi\, e^{i\pi/2}$$

Calculate Product

- $(\cos 3\pi/7 + i\sin 3\pi/7) * (\cos 2\pi/7 + i\sin 2\pi/7)^2$
- Euler's Formula,

---------$(\cos 3\pi/7 + i\sin 3\pi/7) = e^{i3\pi/7}$

---------$(\cos 2\pi/7 + i\sin 2\pi/7)^2 = e^{i4\pi/7}$

$= e^{i3\pi/7} * e^{i2\pi/7}$

$= e^{i(3\pi/7 + 4\pi/7)}$

$= e^{i(7\pi/7)}$

$= e^{i\pi}$

$= \cos\pi + i\sin\pi$

$= -1$

De Moivre’s Theorem

Using Euler's Formula, prove DeMoivre's Theorem.

DeMoivre's Theorem: $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.

Proof (Using Euler's Formula)

Euler's Formula: $\mathrm{e}^{ix} = \cos x + i\sin x$.

Take the case where $x = \theta$.

$e^{i\theta} = \cos\theta + i\sin\theta$

$(\cos\theta + i\sin\theta)^n$

$= (e^{i\theta})^n$

$= e^{i\theta \cdot n}$

$= e^{i(n\theta)}$

$= \cos(n\theta) + i\sin(n\theta)$

Power of a Complex Number

Prove that $(1+i)^4 = -4$

Solution:

As we know, $\theta = \tan^{-1}\left(\frac{y}{x}\right), r = \sqrt{y^2 + x^2}$

Thus, $\theta = \frac{\pi}{4}, r = \sqrt{2}$

Now, apply de Moivre theorem

$$\left(r(\cos\theta + i\sin\theta)\right)^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right)$$

$$\therefore \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)^4 = \left(\sqrt{2}\right)^4\left(\cos(\pi) + i\sin(\pi)\right)$$

$$= 4(-1) = -4$$

As a consequence, $(1+i)^4 = -4$

de Moivre's Theorem: Example

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

let $z = \sqrt{3} + i = 2\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$

then $z^6 = 2^6\left[\cos 6\cdot\frac{\pi}{6} + i\sin 6\cdot\frac{\pi}{6}\right]$

$= 64\left[\cos\pi + i\sin\pi\right]$

$= 64\,[-1 + i\,(0)]$

$$\boxed{z^6 = -64}$$

Find $(1 + i\sqrt{3})^3$ using De Moivre's Theorem.

SOLUTION

Solve Algebraically See Figure 6.63. The argument of $z = 1 + i\sqrt{3}$ is $\theta = \pi/3$, and its modulus is $|1 + i\sqrt{3}| = \sqrt{1 + 3} = 2$. Therefore,

$$z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$z^3 = 2^3\left[\cos\left(3\cdot\frac{\pi}{3}\right) + i\sin\left(3\cdot\frac{\pi}{3}\right)\right]$$

$$= 8(\cos\pi + i\sin\pi)$$

$$= 8(-1 + 0i) = -8$$

Problem 17 : *Determine the values of x and y; if $x + iy = \left(1 + i\sqrt{3}\right)^4$*

Solution :

$$1 + i\sqrt{3} = \sqrt{1+3}\ e^{i\pi/3}$$

$$\therefore \quad (1 + i\sqrt{3})^4 = 2^4 \cdot e^{i4\pi/3}$$

$$= 16\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = 16\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$\therefore \quad x + iy = -8 - i\,8\sqrt{3}$$

$$x = -8 \text{ and } y = -8\sqrt{3}$$

Problem 18 : *Determine the value of $(1 + i)^8 + (1 - i)^8$.*

Solution : $(1 + i)^8 + (1 - i)^8 = \left(\sqrt{2}\, e^{i\pi/4}\right)^8 + \left(\sqrt{2}\, e^{-i\pi/4}\right)^8$

$$= 2^4\left(e^{i2\pi} + e^{-i\,2\pi}\right)$$

$$= 2^4 (\cos 2\pi + i \sin 2\pi + \cos 2\pi - i \sin 2\pi)$$

$$= 2^4 (2 \cos 2\pi)$$

$$= 32$$

Problem 19 : *Determine different values ...*

Complex Numbers
De Moivre's Theorem
Roots of Complex Numbers

Given: The complex number $z = r(\cos\theta + i\sin\theta)$

Prove: $$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

$$k = 0, 1, \cdots, n-1$$

$= 32$

Problem 19 : *Determine different values of the fifth root of $1 + i\sqrt{3}$.* (Apr...

Solution : Let

$$z = 1 + i\sqrt{3}$$

$$= \sqrt{1+3}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$\therefore \quad (1+i\sqrt{3})^{1/5} = 2^{1/5}\left[\cos\left(\frac{\pi/3 + 2k\pi}{5}\right) + i\sin\left(\frac{\pi/3 + 2k\pi}{5}\right)\right]$$

Let z_1, z_2, z_3, z_4 and z_5 be the values of the fifth root of z.

$$z_1 = 2^{1/5}\left[\cos\left(\frac{\pi}{15}\right) + i\sin\left(\frac{\pi}{15}\right)\right] \quad \text{when } k = 0$$

$$z_2 = 2^{1/5}\left[\cos\left(\frac{7\pi}{15}\right) + i\sin\left(\frac{7\pi}{15}\right)\right] \quad \text{when } k = 1$$

$$z_3 = 2^{1/5}\left[\cos\left(\frac{13\pi}{15}\right) + i\sin\left(\frac{13\pi}{15}\right)\right] \quad \text{when } k = 2$$

$$z_4 = 2^{1/5}\left[\cos\left(\frac{19\pi}{15}\right) + i\sin\left(\frac{19\pi}{15}\right)\right] \quad \text{when } k = 3$$

and

$$z_5 = 2^{1/5}\left[\cos\left(\frac{25\pi}{15}\right) + i\sin\left(\frac{25\pi}{15}\right)\right] \quad \text{when } k = 4$$

Thus we find five different values which are complex numbers. If these values a... epresented on the Argand diagram, the corresponding points lie on a circle of radius $2^{1/...}$ he angle between any two succ...

However,

$\therefore$

Problem

Solution

But

$\therefore$

$\therefore$

$\therefore$

The n...

Logarithm of Complex Number

$$\begin{aligned} \text{Log}(z) &= \ln(re^{i\theta}) \\ &= \ln(r) + \ln(e^{i\theta}) \\ &= \ln(r) + i\theta \end{aligned}$$

Where,

r = modulus

Θ = argument

$Z = i$

$Z = x + iy$

Where,

$X = 0, y = 1$

$r = |z| = \sqrt{(x^2 + y^2)} = 1$

$\Theta = \tan^{-1}(y/x) = \tan^{-1}(1/0) = \pi/2$

$Log(z) = \log(i) = \log(1) + \log e^{i\pi/2}$

$Log(i) = 0 + i\pi/2$

$= i\pi/2$

Trigonometric Functions

$$\tan(z) = \frac{Sin(z)}{Cos(z)}$$

$$\cot(z) = \frac{Cos(z)}{Sin(z)}$$

$$\sec(z) = \frac{1}{Cos(z)}$$

$$\csc(z) = \frac{1}{Sin(z)}$$

Relations between cosine, sine and exponential functions

$$e^{\pm i\theta} = \cos(\theta) \pm i \sin(\theta)$$

$$\cos(\theta) = \frac{1}{2}\left(e^{+i\theta} + e^{-i\theta}\right)$$

$$\sin(\theta) = \frac{1}{2i}\left(e^{+i\theta} - e^{-i\theta}\right)$$

Hyperbolic Functions

- **Vincenzo Riccati**
- (1707 - 1775) is given credit for introducing the hyperbolic functions.

Hyperbolic functions are very useful in both mathematics and physics.

Hyperbolic Trigonometric Functions

$$\cosh(ix) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) = \cos x$$
$$\sinh(ix) = \frac{1}{2}\left(e^{ix} - e^{-ix}\right) = i\sin x$$
$$\cosh(x+iy) = \cosh(x)\cos(y) + i\sinh(x)\sin(y)$$
$$\sinh(x+iy) = \sinh(x)\cos(y) + i\cosh(x)\sin(y)$$
$$\tanh(ix) = i\tan x$$
$$\cosh x = \cos(ix)$$
$$\sinh x = -i\sin(ix)$$
$$\tanh x = -i\tan(ix)$$

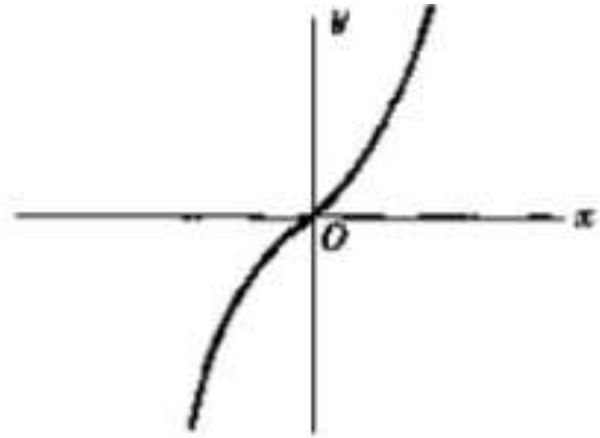
sin hx (hyperbolic of sin x) $= \dfrac{e^x - e^{-x}}{2}$

cos hx (hyperbolic of cos x) $= \dfrac{e^x + e^{-x}}{2}$

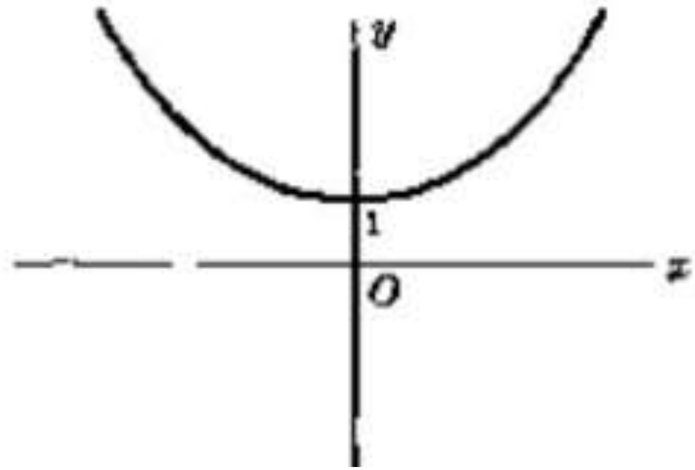
tan hx (hyperbolic of tan x) $= \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Graphs Of Hyperbolic Functions

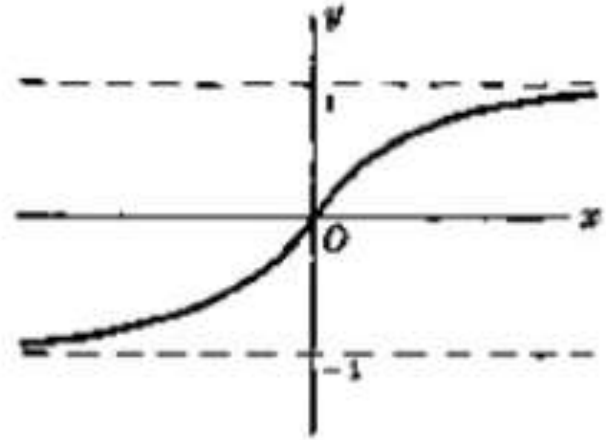
- $y = \sinh x$



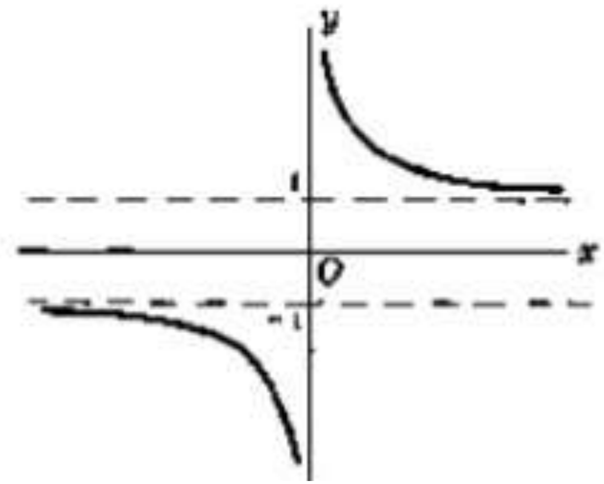
- $y = \cosh x$



- $y = \tanh x$



- $y = \coth x$



Proof of Hyperbolic Identities

- $(\cosh x)^2 - (\sinh x)^2 = 1$

The verification is straightforward:

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{1}{4}\left(\left(e^{2x} + 2e^x e^{-x} + e^{-2x}\right) - \left(e^{2x} - 2e^x e^{-x} + e^{-2x}\right)\right) \\
&= \frac{1}{4}\left(4e^x e^{-x}\right) = 1.
\end{aligned}$$

- $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

Start with the right side and multiply out:

$$
\begin{aligned}
\cosh x \cosh y + \sinh x \sinh y &= \frac{e^x + e^{-x}}{2}\frac{e^y + e^{-y}}{2} \\
&\quad + \frac{e^x - e^{-x}}{2}\frac{e^y - e^{-y}}{2} \\
&= \frac{1}{4}\cdot\Big(\big(e^{x+y} + e^{x-y} + e^{y-x} + e^{-x-y}\big) \\
&\quad + \big(e^{x+y} - e^{x-y} - e^{y-x} + e^{-x-y}\big)\Big) \\
&= \frac{1}{4}\cdot\left(2e^{x+y} + 2e^{-(x+y)}\right) \\
&= \cosh(x+y)
\end{aligned}
$$

- We know that $sinx = \frac{e^{ix} - e^{-ix}}{2i}$

replacing x by ix

$$sin(ix) = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{e^{-x} - e^{x}}{2i}$$

$$= -\frac{e^{x} - e^{-x}}{2i} = i\left(\frac{e^{x} - e^{-x}}{2}\right)$$

$$\therefore sin(ix) = isinh(x)$$

$$\sin^2\theta + \cos^2\theta = \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^2 + \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^2$$

$$= \frac{e^{2i\theta} + e^{-2i\theta} - 2}{-4} + \frac{e^{2i\theta} + e^{-2i\theta} + 2}{4}$$

$$= \frac{-e^{2i\theta} + 2 - e^{-2i\theta} + e^{2i\theta} + e^{-2i\theta} + 2}{4}$$

$\therefore$ $$\sin^2\theta + \cos^2\theta = 1$$

(vii) $$\sin 2\theta = 2\sin\theta\cos\theta$$

$$2\sin\theta\cos\theta = 2\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)$$

$$= \frac{e^{2i\theta} - e^{-2i\theta}}{2i}$$

$$= \frac{e^{i(2\theta)} - e^{-i(2\theta)}}{2i}$$

$\therefore$ $$2\sin\theta\cos\theta = \sin 2\theta$$

(viii) $$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos^2\theta - \sin^2\theta = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^2 - \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^2$$

$$= \frac{e^{2i\theta} + e^{-2i\theta} + 2}{4} - \frac{e^{2i\theta} + e^{-2i\theta} - 2}{-4}$$

$$= \frac{e^{2i\theta} + e^{-2i\theta} + 2 + e^{2i\theta} + e^{-2i\theta} - 2}{4}$$

$$= \frac{e^{i(2\theta)} + e^{-i(2\theta)}}{2}$$

$\therefore$ $$\cos^2\theta - \sin^2\theta = \cos 2\theta$$

Thus, the complex number $(1 - i)$

Problem 23 : *If* $z = \frac{1 + \sqrt{3}\, i}{2}$, *evaluate* z^3.

Solution : The given complex number is

$$z = \frac{1}{2} + \frac{\sqrt{3}}{2} i$$

$$\therefore \quad x = \frac{1}{2} \text{ and } y = \frac{\sqrt{3}}{2}$$

The modulus of a complex number is

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\therefore \quad r = 1$$

The argument of a complex number is

$$\text{Arg}(z) = \theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{\sqrt{3}/2}{\frac{1}{2}}\right) = \tan^{-1}\sqrt{3}$$

$$\therefore \quad \theta = \frac{\pi}{3}$$

Hence, the complex number $z = \frac{1 + i\sqrt{3}}{2}$ in exponential form $= e^{i\pi/3}$.

$$z^3 = (e^{i\pi/3})^3$$

$$= e^{i\pi}$$

$$= \cos\pi + i\sin\pi$$

$$= -1 + 0$$

$$\therefore \quad z^3 = -1$$

Thank You!