Chapter 2

Damped oscillations

Damped oscillations-

When external resistive force due to air or medium acts on an oscillator, the motion of oscillator is reduced that is called damped oscillations. The external resistive force is also called damping force and it is proportional to velocity of oscillator, so two forces are acts on oscillator.

1. Restoring force which is proportional to the displacement and
2. External resistive force proportional to the velocity

Restoring force tries to maintain oscillations while damping force opposes the oscillations, due to this opposition, amplitude of oscillations decreases with time.

$\boldsymbol{Restoring\ force \propto x}$ $\quad f \propto x \ and \ f \propto \frac{dx}{dt}$

$\boldsymbol{Resistive\ force\ \propto velocity}$ $\quad f = -kx \ and \ f = -R\frac{dx}{dT}$

Negative sign indicate that equation shows restoring force is opposite to the **displacement** and **velocity**.

Total force acting on oscillator is,

$$F = -kx - R\frac{dx}{dt}$$

According to Newton's second law of motion,

$$F = m\frac{d^2x}{dt^2}$$

Compare above to equation, we get second order, linear and homogeneous differential equation in x.

$$m\frac{d^2x}{dt^2} + R\frac{dx}{dt} + kx = 0$$

This is different equation of motion for damped oscillations.

Solution of equation of DHO-

Consider $x = ae^{\alpha t}$

Differentiate this equation twice and put values in equation of DHO, We get quadratic equation in α,

$m\alpha^2 + R\alpha + k = 0$, roots of this eq^{n}s. are,

$$\alpha = \frac{-R + \sqrt{R^2 - 4mk}}{2m} \; and \; \frac{-R - \sqrt{R^2 - 4mk}}{2m}$$

So roots are,

$\alpha_1 = \frac{-R+\sqrt{R^2-4mk}}{2m}$, $\alpha_2 = \frac{-R-\sqrt{R^2-4mk}}{2m}$

Thus equation has two solutions $e^{\alpha_1 t} and\ e^{\alpha_2 t}$

Thus solution of DHO is,

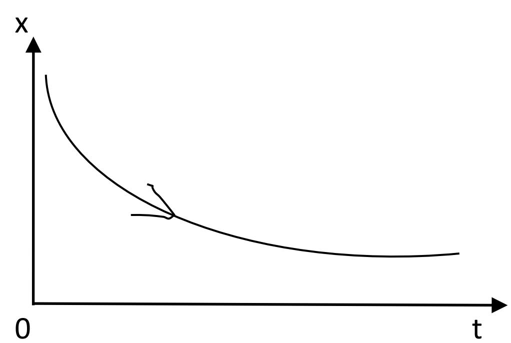
$$x = a_1 e^{(\frac{-R + \sqrt{R^2 - 4mk}}{2m})t} + a_2 e^{(\frac{-R - \sqrt{R^2 - 4mk}}{2m})t}$$

Where, a_1 and a_2 are arbitrary constants.

Different cases-

1. **Damping or Resistive force greater than restoring force,** i.e. $\frac{R^2}{4m^2} > \frac{k}{m}$

Both terms are negative hence both the exponential terms in equation decreases as time increases.



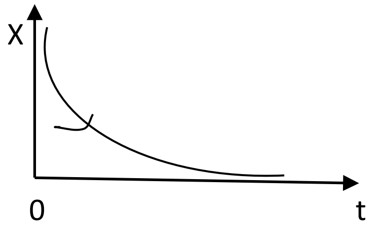
$$x = ae^{-\frac{Rt}{2m}}$$

This is variation of displacement with time. This kind of motion is called **over-damped motion**. For example, oscillation of pendulum moving in thick oil

2. **Damping force is equal to restoring force,** i.e. $\frac{R^2}{4m^2} = \frac{k}{m}$

Here, $\sqrt{\frac{R^2}{4m^2} - \frac{k}{m}}$ is very small, $x = (A + Bt)e^{-\frac{Rt}{2m}}$

This equation represent exponential curve as shown in graph.



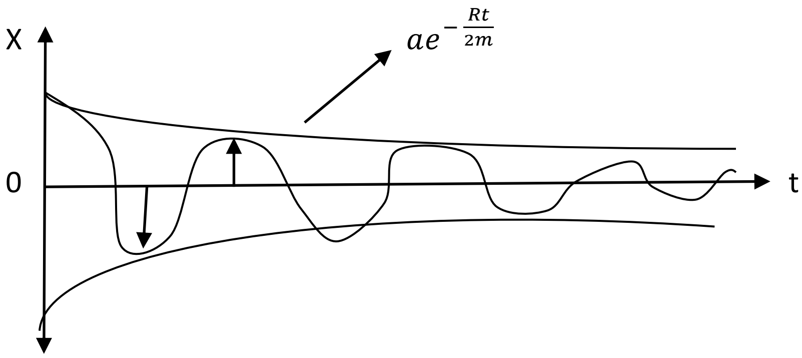
Displacement-time curve for critically damped oscillation, the graph shows it is not oscillatory motion. This is called **critically damped motion.**

3. **Damping force is less than restoring force**, i.e. $\frac{R^2}{4m^2} < \frac{k}{m}$

The term, $\sqrt{\frac{R^2}{4m^2} - \frac{k}{m}}$ becomes imaginary,

$$x = ae^{-\frac{Rt}{2m}}\sin(pt + \theta)$$

This equation gives oscillatory motion with exponentially decreasing amplitude $ae^{-\frac{Rt}{2m}}$ and angular frequency p. This is called damped oscillatory motion.



The frequency of damped oscillation is given as,

$$v_d = \frac{p}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{R^2}{4m^2}}$$

The frequency of un-damped motion is,

$$v = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Therefore, $\boldsymbol{v_d < v}$

Thus the **fall of frequency** and **decrease of amplitude** are the two effect of damping.

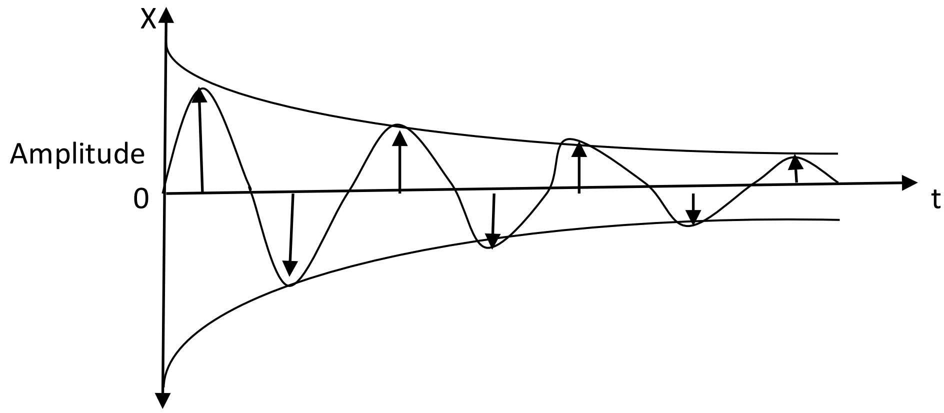
Case	Force condition		Oscillations
1	Damping force > restoring force	$\frac{R^2}{4m^2} > \frac{k}{m}$	Over damped
2	Damping force = restoring force	$\frac{R^2}{4m^2} = \frac{k}{m}$	Critically damped
3	Damping force < restoring force	$\frac{R^2}{4m^2} < \frac{k}{m}$	Under damped

Logarithmic Decrement (λ)-

We know that, equation of damped harmonic oscillator is given as,

$$x = ae^{-\frac{Rt}{2m}}\sin(pt + \theta)$$

Amplitude of oscillation $ae^{-\frac{Rt}{2m}}$ decreases exponentially with time due to damping factor $e^{-\frac{Rt}{2m}}$.



Let us take the successive amplitudes a_1, a_2, a_3, a_4, a_5, ... Let us period of oscillation be T

Amplitude at time t is,

$a_1 = ae^{-\frac{Rt}{2m}}$ Time increased by T for every waveform, i.e. t, t+t/2, t+T, t+3T/2…and taking ratio of amplitudes, we get,

$\frac{a_1}{a_2} = \frac{a_2}{a_3} = \cdots = e^{\frac{RT}{4m}}$ by taking log on both side, exponential term is neglected,

$ln\frac{a_1}{a_2} = ln\frac{a_2}{a_3} = \cdots = \frac{RT}{4m}$

The quantity $\frac{RT}{4m}$ is constant and called logarithmic decrement and denoted by λ.

λ= $\boldsymbol{\frac{RT}{4m}}$

Energy equation of a Damped harmonic oscillator-

$-\frac{dE}{dt} = R\left(\frac{dx}{dt}\right)^2$ The quantity $-\frac{dE}{dt}$ is called dissipation energy. That means amplitude of oscillations die out slowly with time.

Average energy of the oscillator decreases exponentially with time due to resistive force.

$\vec{E} = E_0e^{-\frac{Rt}{m}}$

Power of Damped oscillator-

The **rate of dissipation of energy** is called power dissipation.

P= Damping force x Velocity

Quality factor of DHO-

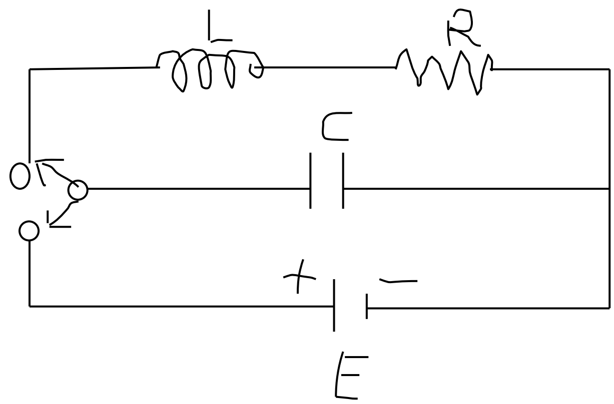
$$Quality\ factor = 2\pi \frac{Energy\ stored\ at\ an\ instant}{Energy\ dissipation\ in\ one\ oscillation}$$

$$Q = 2\pi \frac{\vec{E}}{-T\frac{dE}{dt}}$$

$$Q = \frac{mw}{R}$$

For low value of R, the quality factor Q is large and efficiency of the oscillator is large and vice versa.

Damped oscillations in electrical circuits (LCR series circuit)



Use Kirchhoff's second law,

$V_L + V_R + V_C = 0$

$$V_L = L\frac{dI}{dt}$$

$$V_R = IR$$

$$V_C = \frac{q}{c}$$

This is equation of electrical DHO,

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = 0$$

Solution of this equation is,

$$q = q_1 e^{\left(-\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)t} + q_2 e^{\left(-\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)t}$$

Sr. No.	Condition	Type of discharge
1	$R^2 > \frac{4L}{C}$	Over damped and not oscillatory
2	$R^2 = \frac{4L}{C}$	Critically damped and not oscillatory
3	$R^2 < \frac{4L}{C}$	Under damped and oscillatory