4. Complex Number

A number of the form $z = x + iy$ where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$ with $i^2 = -1$ is called complex number.

For complex number $z = (x, y)$

i) x is called as real part of z $(Re(z))$

ii) y is called imaginary part of z $[Im(z)]$

Remark.

i) A complex number whose real part is zero is called as purely imaginary no.

Notation :- $z = (0, y)$ or $z = iy$

ii) A complex number whose imaginary part is zero is a real number.

$z = (x, 0)$ or $z = x$.

iii) A complex number whose both imaginary and real part is zero is the zero complex number.

$z = 0 + i0$

iv) Two complex number $z_1 = (x_1, y_1)$ & $z_2 = (x_2, y_2)$ are equal if their real part and imaginary part are equal.

i.e $z_1 = z_2$

if $x_1 = x_2$

& $y_1 = y_2$

$z_1 = x_1 + iy_1$
$z_2 = x_2 + iy_2$
$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

$$
\begin{aligned}
z_1 \times z_2 &= (x_1 + iy_1) \times (x_2 + iy_2) \\
&= x_1(x_2 + iy_2) + iy_1(x_2 + iy_2) \\
&= x_1x_2 + ix_1y_2 + iy_1x_2 + (i)^2 y_1y_2 \\
&= x_1x_2 + i(x_1y_2 + iy_1x_2) - y_1y_2 \\
&= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)
\end{aligned}
$$

$Re(z_1z_2) = x_1x_2 - y_1y_2$
$Im(z_1z_2) = x_1y_2 + x_2y_1$

* Basic properties
If z_1, z_2, z_3 are complex number then

i) commutative property.
$z_1 + z_2 = z_2 + z_1$
$z_1z_2 = z_2z_1$

ii) Associative property
$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$
$z_1 (z_2 \cdot z_3) = (z_1 \cdot z_2) z_3$

iii) distributive property
$z_1 (z_2 + z_3) = z_1z_2 + z_1z_3$

iv) Additive and multiplicative identity.

The complex no. zero $= (0,0)$ is additive identity of z and $1 = (1,0)$ is multiplicative identity of z.

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i.e. $z+0=z$

$z\cdot 1=z$

Additive and multiplicative inverse.

For complex number $z=(x,y)$ is ~~mul~~ additive inverse equal to $-z=(-x,-y)$

For any non-zero complex number $zz_1 = z_1z = 1$ & If $z=(x,y)$

$$z_1 = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

1. Show that $1+z^2 = 1+2z+z^2$

→ by distributive property

$1+z^2 = (1+z)(1+z)$

$1+z^2 = 1+2z+z^2$

2. Verify that each of the two no. $z = 1\pm i$ satisfying the equation $z^2-2z+2=0$

→ If $z=1\pm i$ then z^2-2z+2

$= (1\pm i)^2 - 2(1\pm i)+2$

$= \pm 2i - 2+2i+2$

$= 0.$

3 Nov-2025

IMP
5 marks

De-Moivre's theorem

Let θ be the any real number and n be an integer then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Proof:-

We prove the theorem only for integral value of n in three cases

Let $z = \cos\theta + i\sin\theta$

we can write these as

$z = e^{i\theta}$

i.e we have to prove that

$z^n = e^{in\theta}$

case i)

Let n be the positive integer.

In this case we prove the theorem by mathematical induction.

Let $P(n) = z^n = e^{in\theta}$

step I) If $n=1$, $z^1 = e^{i\theta}$ ie $z = e^{i\theta}$

$\therefore P(1)$ is true for $n=1$

Step II

Let us assume that

$P(m)$ is true for positive value of m

$z^m = e^{im\theta}$

step III

Now, we want to show that

$P(m+1)$ is true.

$z^{m+1} = z^m . z^1$

$= e^{im\theta} . e^{i\theta}$

$= e^{im\theta + i\theta}$

$= e^{i\theta(m+1)}$

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∴ p(m+1) is true.
∴ By mathematical induction,
p(n) is true for all positive integer.

Case ii)
Let $n = 0$
Then $z^0 = 1 = \cos 0\theta + i \sin 0\theta$
$= 1 + 0$
$= 1$

∴ Theorem is true for $n=0$.

Case iii)
Let n be the negative integer ~~th~~
then their is positive integer m such that
$n = -m$
$\therefore z^n = z^{-m} = \frac{1}{z^m} = \frac{1}{e^{im\theta}} = e^{-im\theta} = e^{in\theta}$

Thus theorem is also true for negative integer m.

marks Q IMP Find expression for $\cos^6\theta$ and $\sin^6\theta$ in term of cosin and sin of multiple of θ.
→ Let $z = \cos\theta + i\sin\theta$
by de-moivre's thm
For any integer $k>0$ we have
$z^k = \cos k\theta + i \sin k\theta$
$z^{-k} = \cos(-k)\theta + i\sin(-k)\theta$
$z^{-k} = \cos k\theta - i \sin k\theta$ ∵ $\cos(-\theta) = \cos\theta$
$\sin(-\theta) = -\sin\theta$
so that
$z^k + z^{-k} = 2\cos k\theta$
$z^k - z^{-k} = 2i\sin k\theta$.

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For $k=1$, $z+z^{-1}=z+\frac{1}{z}=2\cos\theta$

$$\left(z+\frac{1}{z}\right)^6=\binom{6}{0}z^6\left(\frac{1}{z}\right)^0+\binom{6}{1}z^5\left(\frac{1}{z}\right)^1+\binom{6}{2}z^4\left(\frac{1}{z}\right)^2+\binom{6}{3}z^3\left(\frac{1}{z}\right)^3$$
$$+\binom{6}{4}z^2\left(\frac{1}{z}\right)^4+\binom{6}{5}z^1\left(\frac{1}{z}\right)^5+\binom{6}{6}z^0\left(\frac{1}{z}\right)^6$$

$$=z^6+6z^5\left(\frac{1}{z}\right)+15z^4\left(\frac{1}{z^2}\right)+20z^3\left(\frac{1}{z^3}\right)+$$
$$15z^2\left(\frac{1}{z^4}\right)+6z\left(\frac{1}{z^5}\right)+\left(\frac{1}{z^6}\right)$$

$$=z^6+6z^4+15z^2+20+15\left(\frac{1}{z^2}\right)+6\left(\frac{1}{z^4}\right)+\left(\frac{1}{z^6}\right)$$

$$2^6\cos^6\theta=1\left(z^6+\frac{1}{z^6}\right)+6\left(z^4+\frac{1}{z^4}\right)+15\left(z^2+\frac{1}{z^2}\right)+20$$

$$\cos^6\theta=\frac{1}{32}\left(z^6+\frac{1}{z^6}\right)+\frac{6}{32}\left(z^4+\frac{1}{z^4}\right)+\frac{15}{32}\left(z^2+\frac{1}{z^2}\right)+\frac{20}{32}$$

$$\cos^6\theta=\frac{1}{32}\cos 6\theta+\frac{3}{16}\cos 4\theta+\frac{15}{32}\cos 2\theta$$
$$+\frac{5}{8}$$

Ily
We have for $k=1$, $z-z^{-1}=z-\frac{1}{z}=2i\sin\theta$

$$2^6(i)^6\sin^6\theta=\binom{6}{0}z^6\left(\frac{1}{z}\right)^0-\binom{6}{1}z^5\left(\frac{1}{z}\right)^1+\binom{6}{2}z^4\left(\frac{1}{z}\right)^2$$
$$-\binom{6}{3}z^3\left(\frac{1}{z}\right)^3+\binom{6}{4}z^2\left(\frac{1}{z}\right)^4-\binom{6}{5}z^1\left(\frac{1}{z}\right)^5+\binom{6}{6}z^0\left(\frac{1}{z}\right)^6$$

$$-2^6 \sin^6\theta = z^6 - 6z^5\left(\frac{1}{z}\right) + 15z^4\left(\frac{1}{z^2}\right) - 20z^3\left(\frac{1}{z^3}\right) + 15z^2\left(\frac{1}{z^4}\right) - 6z\left(\frac{1}{z^5}\right) + \left(\frac{1}{z^6}\right)$$

$$-2^6 \sin^6\theta = z^6 - 6z^4 + 15z^2 - 20 + 15\left(\frac{1}{z^2}\right) - 6\left(\frac{1}{z^4}\right) + \frac{1}{z^6}$$

$$-2^6 \sin^6\theta = 1\left(z^6 + \frac{1}{z^6}\right) - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20$$

$$-2^6 \sin^6\theta = 1\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 20$$

$$\sin^6\theta = -\frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta - \frac{15}{32}\cos 2\theta + \frac{5}{8}$$

- Find the value of $\bar{z}$ & $|z|$ where $z = 3+4i$

1 marks

→ Let $z = 3+4i$

$\bar{z} = 3-4i$

& $|z| = \sqrt{(3)^2+(4)^2} = \sqrt{9+16} = \sqrt{25} = 5$

$|z| = 5$

Find the value $\overline{17}$ Z_5 Find the value of $\overline{17}$ in Z_5

We know that,

$Z_5 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$

$\therefore 17 \equiv 2 \pmod 5$

$\therefore$ value of $\overline{17}$ in Z_5 is $\bar{2}$.

$\overline{14}$	Z_3
$\overline{18}$	Z_4

Bolzano's thm

Statement: Let $f: I \to \mathbb{R}$ be continuous on $I = [a, b]$ If $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$ then prove that their exist a point c, $c \in I$ a & b show that $f(c) = k$.

→ proof: suppose $a < b$

$$g(x) = f(x) - k$$
$$g(a) = f(a) - k \text{ \& } g(b) = f(b) - k$$

Now,

$$f(a) < k < f(b)$$
$$f(a) - k < k - k < f(b) - k$$
$$g(a) < 0 < g(b)$$
$$\therefore \exists\, c \in (a, b) \quad \text{s.t. } f(a) < 0 < f(b)$$
$$g(c) = 0 \qquad \therefore g(c) = f(c) - k)$$
$$\text{i.e. } f(c) - k = 0$$
$$f(c) = k$$

Hence proved.

Statement: Let $A \subseteq \mathbb{R}$, $f, g: A \to \mathbb{R}$ if f & g are continuous function at $c \in A$ then prove that $f+g$ & $f-g$ also conti function at $x = c$.

proof →

f & g are continuous at $x = c$

$$\therefore \lim_{x \to c} f(x) = f(c) \text{ \& } \lim_{x \to c} g(x) = g(c)$$

Consider

$$\lim_{x \to c} [f(g)(x)] = \lim_{x \to c} [f(x) + g(x)]$$
$$= \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$$
$$= f(c) + g(c)$$
$$= (f+g)(c)$$

$= f+g$ is continuous

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Now $\lim_{x \to c} [(f \cdot g)(x)] = \lim_{x \to c} [f(x) \cdot g(x)]$

$= \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$

$= f(c) \cdot g(c) = (f \cdot g) c$

$\therefore$ $f \cdot g$ is continuous