K. T. S. P. Mandal's

Hutatma Rajguru Mahavidyalaya, Rajgurunagar

Tal-Khed, Dist-Pune (410 505)

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Subject – Vector Calculus

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Chapter 2: Integrals

Topic- Conservative Field

Prepared by

Prof. R. M. Wayal

Department of Mathematics

Hutatma Rajguru Mahavidyalaya, Rajgurunagar

HRMMATHS

* Path Independent and conservative -

Let $\bar{F}$ be a vector field defined on an open region D in space, and suppose that for every two points A and B in D the line integral

$$\int_C \bar{F} \cdot d\bar{r}$$

along a path C from A to B in D is the same over all paths from A to B. Then the integral $\int_C \bar{F} \cdot d\bar{r}$ is path independent in D and the field $\bar{F}$ is conservative on D.

* Potential Function -

Let $\bar{F}$ be a vector field defined on D and $\bar{F} = \nabla f$ for some scalar function f on D. then f is called a potential function of $\bar{F}$

$\bar{F}$ = conservative field
f = potential f^n

Thm -
Let C be a smooth curve joining the point A to the point B in the plane or in the space and parametrized by $\bar{r}(t)$. Let f be a differentiable f^n with continuous gradient vector $\bar{F} = \nabla f$ on a domain D containing C. Then

$$\boxed{\int_C \bar{F} \cdot d\bar{r} = f(B) - f(A)}$$

Thm - conservative fields are gradients field -
Let $\bar{F} = M\bar{i} + N\bar{j} + P\bar{k}$ be a vector field whose components are continuous throughout an open connected region D in space. Then $\bar{F}$ is conservative if & only if $\bar{F}$ is gradient

field ∇f for a diff. f^n f.

i.e. $\bar{F} = \nabla f$ iff $\bar{F}$ is conservative

[f - potential f^n]

Thm - Loop property of conservative field -

The following statements are equivalent.

1) $\oint_C \bar{F} \cdot d\bar{r} = 0$ around every loop (closed curve C) in D.

2) The field $\bar{F}$ is conservative on D.

Remark - $\bar{F}$ is conservative iff

1) $\int_C \bar{F} \cdot d\bar{r}$ does not depends on path

2) $\bar{F} = \nabla f$

3) $\oint_C \bar{F} \cdot d\bar{r} = 0$

* Component Test for conservative Fields -

Let $\bar{F} = M(x,y,z)\,\bar{i} + N(x,y,z)\,\bar{j} + P(x,y,z)\,\bar{k}$ be a field on a connected and simply connected domain whose component function have conti first partial derivatives. Then $\bar{F}$ is conservative if and only if

$$\frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}, \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}$$

[$M \to x$, $N \to y$, $P \to z$]

* Exact Diff. Eq^n -

$M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz$ is said to be differential if

$$M\,dx + N\,dy + P\,dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df$$

* Component Test for Exactness of $M\,dx + N\,dy + P\,dz$ is

$$\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \ \& \ \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}$$

* If $M\,dx + N\,dy + P\,dz = 0$ then soln is

$$\underset{y,z-\text{const}}{\int M\,dx} + \underset{z-\text{const}}{\int}(\text{term in } N \text{ not containing } x)\,dy$$
$$+ \int(\text{term in } P \text{ not containing } x \ \& \ y)\,dz = C$$

Ex - Determine whether the vector field is conservative or not

$$\bar{F} = y\sin z\,\bar{i} + x\sin z\,\bar{j} + xy\cos z\,\bar{k}$$

$\Rightarrow$ Let $M = y\sin z$, $N = x\sin z$, $P = xy\cos z$

$M \to x,\ N \to y,\ P \to z$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
$$\frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}$$
$$\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$$

$$\frac{\partial M}{\partial y} = \sin z \ \& \ \frac{\partial N}{\partial x} = \sin z$$
$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial N}{\partial z} = x\cos z \ \& \ \frac{\partial P}{\partial y} = x\cos z$$
$$\Rightarrow \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}$$

$$\frac{\partial M}{\partial z} = y\cos z \ \& \ \frac{\partial P}{\partial x} = y\cos z$$
$$\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$$

$\therefore \bar{F}$ is is conservative

Remark - If $\bar{F} = M(x,y)\,\bar{i} + N(x,y)\,\bar{j}$ (or $M\,dx + N\,dy$) is conservative iff
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Ex. Determine whether the vector field conservative or not.

$$\bar{F}(x,y) = (x^2 - yx)\bar{i} + (y^2 - xy)\bar{j}$$

$\Rightarrow$ $M = x^2 - yx$ & $N = y^2 - xy$

$$\frac{\partial M}{\partial y} = -x \quad \& \quad \frac{\partial N}{\partial x} = -y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ $\bar{F}$ is not conservative.

Ex. $\bar{F}(x,y) = (2x e^{xy} + x^2 y e^{xy})\bar{i} + (x^3 e^{xy} + 2y)\bar{j}$

$\Rightarrow$ $M = 2x e^{xy} + x^2 y e^{xy}$ & $N = x^3 e^{xy} + 2y$

$$\frac{\partial M}{\partial y} = 2x e^{xy}(x) + x^2 (1) e^{xy} + x^2 y e^{xy} \cdot x$$

$$= e^{xy}[2x^2 + x^2 + x^3 y]$$

$$= e^{xy}[3x^2 + x^3 y]$$

$$\frac{\partial N}{\partial x} = x^3 e^{xy}(y) + 3x^2 e^{xy}$$

$$= e^{xy}[x^3 y + 3x^2]$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ $\bar{F}$ is conservative.

Ex. Find the potential function of the vector field $\bar{F} = 2xy^3z^4\,\bar{i} + 3x^2y^2z^4\,\bar{j} + 4x^2y^3z^3\,\bar{k}$

$\Rightarrow$ $M = 2xy^3z^4$, $N = 3x^2y^2z^4$, $P = 4x^2y^3z^3$

Potential function f is

$$f(x,y,z) = \int_{y,z-\text{const}} M\,dx + \int_{z-\text{const}} (\text{terms in } N \text{ free from } x)\,dy + \int (\text{terms in } P \text{ free from } x,y)\,dz + c$$

$$= \int_{y,z-\text{const}} 2xy^3z^4\,dx + \int_{z-\text{const}} 0\,dy + \int 0\,dz + c$$

$$= 2y^3z^4\,\frac{x^2}{2} + c$$

$$f(x,y,z) = x^2y^3z^4 + c$$

Ex. Find potential function for the vector field

$$\bar{F} = (2x\cos y - 2z^3)\bar{i} + (3 + 2ye^z - x^2\sin y)\bar{j} + (y^2e^z - 6xz^2)\bar{k}$$

$$\Rightarrow M = 2x\cos y - 2z^3, \quad N = 3 + 2ye^z - \cancel{x^2\sin y}$$

$$P = y^2\cancel{e^z} - \cancel{6xz^2}$$

$$f(x,y,z) = \int_{y,z-\text{const}} M\,dx + \int_{z-\text{const}} [\text{terms in } N \text{ free from } x]\,dy + \int (\text{terms } P \text{ free from } x,y)\,dz + c$$

$$= \int_{y,z-\text{const}} [2x\cos y - 2z^3]\,dx + \int_{z-\text{const}} (3 + 2ye^z)\,dy + \int 0\,dz + c$$

$$= 2\cos y\cdot\frac{x^2}{2} - 2z^3x + 3y + 2e^z\,\frac{y^2}{2} + c$$

$$f(x,y,z) = x^2\cos y - 2xz^3 + 3y + y^2e^z + c$$

Ex. Find the work done by the conservative field $\bar{F} = yz\bar{i} + xz\bar{j} + xy\bar{k}$ where $f(x,y,z) = xyz$ in moving an object along any smooth curve c joining the point $A(-1,3,9)$ to $B(1,6,-4)$

$$\Rightarrow \text{Work done} = \int_c \bar{F}\cdot d\bar{r}$$

$$= f(B) - f(A)$$

$f(x,y,z) = xyz$

$$= f(1,6,-4) - f(-1,3,9)$$

$$= 1(6)(-4) - (-1)(3)(9)$$

$$= -24 + 27$$

$$= 3$$

Ex. suppose the force field $\bar{F} = \nabla f$ is the gradient of the function $f(x,y,z) = \dfrac{-1}{x^2+y^2+z^2}$

Find the work done by $\bar{F}$ in moving an object along a smooth curve c joining $(1,0,0)$ to $(0,0,2)$ that does not pass through the origin.

$\Rightarrow$ Work done $= \int_c \bar{F} \cdot d\bar{r}$

$$= f(B) - f(A)$$

$$= f(0,0,2) - f(1,0,0)$$

$$= \frac{-1}{4} + \frac{1}{1}$$

$$= \frac{3}{4}$$

Ex. Find a potential function for the field and evaluate the integral

i) $$\int_{(1,1,1)}^{(1,2,3)} 3x^2 dx + \frac{z^2}{4} dy + 2z \log y \, dz$$

$\Rightarrow$ Let $f(x,y,z)$ be the potential function of the field

$$\& \; M = 3x^2, \quad N = \frac{z^2}{4}, \quad P = 2z \log y$$

$$f(x,y,z) = \int_{y,z-\text{const}} M dx + \int_{z-\text{const}} (\text{terms in } N \text{ free from } x) dy$$

$$+ \int (\text{terms in } z \text{ free from } x,y) dz + c$$

$$= \int 3x^2 dx + \int \underset{z\text{-const}}{\frac{z^2}{y}} dy + \int 0 dz + c$$

$$f(x,y,z) = x^3 + z^2 \log y + c$$

$$\therefore \int_{(1,1,1)}^{(1,2,3)} 3x^2 dx + \frac{z^2}{y} dy + 2z \log y \, dz = f(1,2,3) - f(1,1,1)$$

$$= [1 + 9\log 2 + c] - [1 + \log 1 + c]$$

$$= 9\log 2$$

ii) $\displaystyle\int_{(1,2,1)}^{(2,1,1)} (2x\log y - yz)dx + \left(\frac{x^2}{y} - xz\right)dy - xy\,dz$

$$\Rightarrow f(x,y,z) = \int \underset{y,z\text{-const}}{[2x\log y - yz]} dx + \int 0 dy + \int 0 dz + c$$

$$= 2\log y \cdot \frac{x^2}{2} - xyz + c$$

$$f(x,y,z) = x^2 \log y - xyz + c$$

$$\int_{(1,2,1)}^{(2,1,1)} (2x\log y - yz)dx + \left(\frac{x^2}{y} - xz\right)dy - xy\,dz$$

$$= f(2,1,1) - f(1,2,1)$$

$$= [4\log 1 - 2 + c] - [1\log 2 - 2 + c]$$

$$= 0 - 2 + c - \log 2 + 2 - c$$

$$= -\log 2$$

iii) $\displaystyle\int_{(1,1,1)}^{(2,2,2)} \frac{1}{y} dx + \left(\frac{1}{z} - \frac{x}{y^2}\right)dy - \frac{y}{z^2} dz$

$$\Rightarrow f(x,y,z) = \int \underset{y,z\text{-const}}{\frac{1}{y}} dx + \int \underset{z\text{-const}}{\frac{1}{z}} dy - \int 0 dz + c$$

$$= \frac{x}{y} + \frac{y}{z} + c$$

$$\int_{(1,1,1)}^{(2,2,2)} \frac{1}{y}dx + \left(\frac{1}{z} - \frac{x}{y^2}\right)dy - \frac{y}{z^2}dz = f(2,2,2) - f(1,1,1)$$

$$= (1+1+c) - (1+1+c)$$

$$= 0$$

iv) $$\int_{(-1,-1,-1)}^{(2,2,2)} \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2}$$

$$\Rightarrow f(x,y,z) = \int_{y,z-\text{const}} \frac{2x}{x^2+y^2+z^2}dx + \int 0\,dy + \int 0\,dz + c$$

$$= \log(x^2+y^2+z^2) + c$$

$$\int_{(-1,-1,-1)}^{(2,2,2)} \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2} = f(2,2,2) - f(-1,-1,-1)$$

$$= \log(12) - \log(3)$$

$$= \log\left(\frac{12}{3}\right)$$

$$= \log 4$$

Refrence - A text book of S.Y.B.Sc. vector calculus by Golden series, Nirali Publication.