K. T. S. P. Mandal's

Hutatma Rajguru Mahavidyalaya, Rajgurunagar

Tal-Khed, Dist-Pune (410 505)

Second Year Science

Semester -IV (2019 Pattern)

Subject – Vector Calculus

S. Y. B. Sc., Paper-II:MT-242(A)

Chapter 2: Integrals

Topic- Green's Theorem

Prepared by

Prof. R. M. Wayal

Department of Mathematics

Hutatma Rajguru Mahavidyalaya, Rajgurunagar

Defⁿ – Circulation density or the $\bar{k}$ – component of the curl –

The circulation density of a vector field $\bar{F} = M\bar{i} + N\bar{j}$ at the point (x, y) is the scalar expression

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

This expression is also called $\bar{k}$ – component of curl. & denoted by $(\text{curl } \bar{F}) \cdot \bar{k}$

* Divergence –

The divergence (flux density) of a vector field $\bar{F} = M\bar{i} + N\bar{j}$ at the point (x, y) is

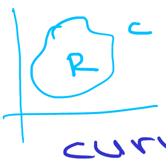
$$\text{div } \bar{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

Remark –

1) If $\text{div } \bar{F}(x_0, y_0) > 0$ then a gas expanding at the point (x_0, y_0)

2) If $\text{div } \bar{F}(x_0, y_0) < 0$ then a gas compressing at the point (x_0, y_0)

* Green's theorem (Circulation – curl or Tangential Form) –



Let c be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\bar{F} = M\bar{i} + N\bar{j}$ be a vector field with M & N having continuous first partial derivative in an open region containing R then

$$\oint_C \bar{F} \cdot \bar{T} \, ds = \oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

Theorem – Green's thm (Flux – Divergence or Normal Form)

Let c be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let

$\bar{F} = M\bar{i} + N\bar{j}$ be a vector field with M & N having continuous first order partial derivatives in an open region containing R. Then

$$\oint_C \bar{F}\cdot\bar{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dx\,dy$$

Ex. Find the divergence and interpret what it means, for each vector field representing the velocity of a gas flowing in the xy-plane.

1) $\bar{F}(x,y) = cx\bar{i} + cy\bar{j}$, c is const.

$\Rightarrow$ $$\operatorname{div}\bar{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

Here $M = cx$ & $N = cy$

$$\therefore \operatorname{div}\bar{F} = c + c = 2c$$

If $c > 0$ then gas is undergoing uniform expansion

If $c < 0$ then gas is undergoing uniform compression.

2) $\bar{F} = -cy\bar{i} + cx\bar{j}$

$\Rightarrow$ $M = -cy$ & $N = cx$

$$\frac{\partial M}{\partial x} = 0 \quad \& \quad \frac{\partial N}{\partial y} = 0$$

$$\operatorname{div}\bar{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0$$

$\therefore$ The gas is neither expanding nor compressing

3) $$\bar{F} = \frac{-y}{x^2+y^2}\bar{i} + \frac{x}{x^2+y^2}\bar{j}$$

$$\frac{u}{v} = \frac{vu' - uv'}{v^2}$$

$\Rightarrow$ $$M = \frac{-y}{x^2+y^2} \quad \& \quad N = \frac{x}{x^2+y^2}$$

$$\frac{\partial M}{\partial x} = \frac{(x^2+y^2)(0) - (-y)2x}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial y} = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\operatorname{div} \bar{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} = 0$$

$\therefore$ The gas is neither expansion nor compression

Ex. Find $\bar{k}$-component of curl$(\bar{F})$ for the vector field $\bar{F} = (x^2-y)\bar{i} + y^2\bar{j}$ on the plane

$\Rightarrow \quad M = x^2 - y$ & $N = y^2$

$$\frac{\partial M}{\partial y} = -1 \quad \& \quad \frac{\partial N}{\partial x} = 0$$

$$\operatorname{curl}(F)\cdot\bar{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$= 0 - (-1) = 1$$

ii) $\bar{F} = xe^y\,\bar{i} + ye^x\,\bar{j}$

$\Rightarrow \quad M = xe^y$ & $N = ye^x$

$$\frac{\partial M}{\partial y} = xe^y \quad \& \quad \frac{\partial N}{\partial x} = ye^x$$

$\bar{k}$-component of curl$(\bar{F})$ is

$$\operatorname{curl}(F)\cdot\bar{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$$= ye^x - xe^y$$

iii) $\bar{F} = (x^2y)\bar{i} + (y^2x)\bar{j}$

$\Rightarrow \quad M = x^2y$ & $N = y^2x$

$$\frac{\partial M}{\partial y} = x^2 \quad \& \quad \frac{\partial N}{\partial x} = y^2$$

$$\operatorname{curl}(\bar{F})\cdot\bar{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = y^2 - x^2$$

Ex. Verify $\oint_C \bar{F}\cdot\bar{T}\,ds = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$

for the vector field $\bar{F} = 2x\bar{i} - 3y\bar{j}$. The region $R: x^2+y^2 \le a^2$ and its bounding circle

$C: \bar{r} = a\cos t\,\bar{i} + a\sin t\,\bar{j}, \quad 0 \le t \le 2\pi$

$\Rightarrow$ 1st solve $\oint_C M\,dx + N\,dy$

$\bar{F} = 2x\,\bar{i} - 3y\,\bar{j}$ & $\bar{r} = a\cos t\,\bar{i} + a\sin t\,\bar{j}$

$\Rightarrow M = 2x$, $N = -3y$, $x = a\cos t$ & $y = a\sin t$

$\Rightarrow dx = -a\sin t\,dt$

& $dy = a\cos t\,dt$

$$\oint_C M\,dx + N\,dy = \oint_C 2x\,dx - 3y\,dy$$

$$= \int_0^{2\pi} 2a\cos t\,(-a\sin t)\,dt - 3a\sin t\,a\cos t\,dt$$

$$= \int_0^{2\pi} [-2a^2\cos t\sin t - 3a^2\cos t\sin t]\,dt$$

$$= -\frac{5a^2}{2}\int_0^{2\pi} 2\cos t\sin t\,dt$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$= -\frac{5}{2}a^2\int_0^{2\pi}\sin 2t\,dt$$

$$= -\frac{5}{2}a^2\left[-\frac{\cos 2t}{2}\right]_0^{2\pi}$$

$$= \frac{5}{4}a^2[\cos 4\pi - \cos 0]$$

$$= \frac{5}{4}a^2[1-1] = 0 \quad \text{--- (1)}$$

Now solve $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$

$M = 2x$, $N = -3y$

$\frac{\partial M}{\partial y} = 0$ & $\frac{\partial N}{\partial x} = 0$

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy = \iint_R 0\,dx\,dy = 0 \quad \text{--- (2)}$$

From (1) & (2)

$\oint \bar{F}\cdot\bar{dr} = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$ is verified.

Ex. Verify

$$\oint_C \bar{F}\cdot\bar{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dx\,dy$$

for the vector field $\bar{F} = y\bar{i}$. The region is $R: (x^2+y^2) \le a^2$ and its bounding circle C is $\bar{r} = a\cos t\,\bar{i} + a\sin t\,\bar{j}$, $0 \le t \le 2\pi$.

$\Rightarrow$ ① $\oint_C M\,dy - N\,dx$

$\bar{F} = y\bar{i}$, $\bar{r} = a\cos t\,\bar{i} + a\sin t\,\bar{j}$

$\therefore M = y$ & $N = 0$

$x = a\cos t$ & $y = a\sin t$

$dx = -a\sin t\,dt$ & $dy = a\cos t\,dt$

$$\oint_C M\,dy - N\,dx = \oint_C y\,dy - 0\,dx$$

$$= \int_0^{2\pi} a\sin t \cdot a\cos t\,dt$$

$$= \frac{a^2}{2}\int_0^{2\pi} 2\sin t\cos t\,dt$$

$$= \frac{a^2}{2}\int_0^{2\pi} \sin 2t\,dt$$

$$= \frac{a^2}{2}\left[-\frac{\cos 2t}{2}\right]_0^{2\pi}$$

$$= -\frac{a^2}{4}\left[\cos 4\pi - \cos 0\right]$$

$$\oint_C M\,dy - N\,dx = -\frac{a^2}{4}[1-1] = 0 \quad \text{—— ①}$$

② $\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dx\,dy$

$M = y$ & $N = 0$

$\frac{\partial M}{\partial x} = 0$ & $\frac{\partial N}{\partial y} = 0$

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R (0+0)\,dx\,dy = 0 \quad \text{——} ②$$

From ① & ②

$$\oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy.$$

Ex. Use Green's thm to find the counterclockwise circulation and outword flux for the field $\bar{F} = (x^2+4y)\bar{i} + (x+y^2)\bar{j}$, where c is the square bounded by $x=0, x=1, y=0, y=1$

$$\Rightarrow \bar{F} = (x^2+4y)\bar{i} + (x+y^2)\bar{j}$$

$$M = x^2+4y \quad \& \quad N = x+y^2$$

① circulation $= \oint_C \bar{F}\cdot\bar{T}\,ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$\frac{\partial M}{\partial y} = 4 \quad \& \quad \frac{\partial N}{\partial x} = 1$$

$$\text{circulation} = \iint_R (1-4)\,dx\,dy$$

$$= -3 \iint_R dx\,dy$$

$$= -3 \int_{y=0}^{1}\int_{x=0}^{1} dx\,dy$$

$$= -3 \int_{y=0}^{1} [x]_0^1\,dy$$

$$= -3 \int_{y=0}^{1} (1-0)\,dy$$

$$= -3\,[y]_0^1$$

$$= -3(1-0) = -3$$

②

$$\text{Flux} = \oint_C \bar{F}\cdot\bar{n}\,ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$$

$$= \iint_R (2x + 2y)\,dx\,dy$$

$$= \int_{y=0}^{1}\int_{x=0}^{1} (2x+2y)\,dx\,dy$$

$$= \int_{y=0}^{1} \left[2\frac{x^2}{2} + 2yx\right]_{x=0}^{1} dy$$

$$= \int_{y=0}^{1} [(1+2y) - 0]\,dy$$

$$= \left[y + \cancel{2}\frac{y^2}{\cancel{2}}\right]_{y=0}^{1}$$

$$= (1+1) - 0$$

$$\text{Flux} = 2$$

Ex. Use Green's thm to evaluate

$\oint_C (2x+y^2)\,dx + (2xy+3y)\,dy$ where c is any simple closed curve in the plane for which Green's thm hold.

$$\Rightarrow \quad \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Here $M = 2x + y^2$ & $N = 2xy + 3y$

$$\frac{\partial M}{\partial y} = 2y \quad \& \quad \frac{\partial N}{\partial x} = 2y$$

$$\oint_C M\,dx + N\,dy = \iint_R [2y - 2y]\,dx\,dy = 0$$

Ex. Use Green's thm to evaluate

$I = \oint_C (3y\,dx + 2x\,dy)$ where c is the boundary of

$0 \le x \le \pi$ & $0 \le y \le \sin x$

$$\Rightarrow \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$M = 3y$ & $N = 2x$

$\frac{\partial M}{\partial y} = 3$ & $\frac{\partial N}{\partial x} = 2$

$$\therefore \quad I = \iint_R (2-3)\,dx\,dy$$

$$= -\iint_R 1\,dx\,dy$$

$$= -\int_{x=0}^{\pi} \int_{y=0}^{\sin x} 1\,dy\,dx$$

$$= -\int_{x=0}^{\pi} [y]_{y=0}^{\sin x}\,dx$$

$$= -\int_{x=0}^{\pi} [\sin x - 0]\,dx$$

$$= -\int_{x=0}^{\pi} \sin x\,dx \quad = -[-\cos x]_0^{\pi}$$

$$= \cos\pi - \cos 0$$

$$= -1 - 1 = -2$$

Ex $I = \oint_C y^3\,dx - x^3\,dy$ where c is positively oriented circle of radius 2 center at the origin. [i.e. R is $x^2+y^2=4$ / $x^2+y^2=2^2$]

$$\Rightarrow \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$M = y^3$ & $N = -x^3$

$$\frac{\partial M}{\partial y} = 3y^2 \quad \& \quad \frac{\partial N}{\partial x} = -3x^2$$

$$\therefore \quad I = \iint_R (-3x^2 - 3y^2)\, dx\, dy$$

$$= -3 \iint_R (x^2+y^2)\, dx\, dy$$

since R is $x^2+y^2 = 2^2$

$\therefore$ put $x = r\cos\theta$, $y = r\sin\theta$

$dx\, dy = r\, dr\, d\theta$

$0 \le r \le 2 \quad \& \quad 0 \le \theta \le 2\pi$

$$\therefore \quad I = -3 \int_{\theta=0}^{2\pi} \int_{r=0}^{2} [r^2\cos^2\theta + r^2\sin^2\theta]\, r\, dr\, d\theta$$

$$= -3 \int_{\theta=0}^{2\pi} \int_{r=0}^{2} r^2 . r\, dr\, d\theta$$

$$= -3 \int_{\theta=0}^{2\pi} \left[\frac{r^4}{4}\right]_0^2 d\theta$$

$$= -\frac{3}{4} \int_{\theta=0}^{2\pi} [16-0]\, d\theta$$

$$= -\frac{3}{\cancel{4}_1} (\cancel{16}^{4}) [\theta]_0^{2\pi}$$

$$= -12(2\pi - 0)$$

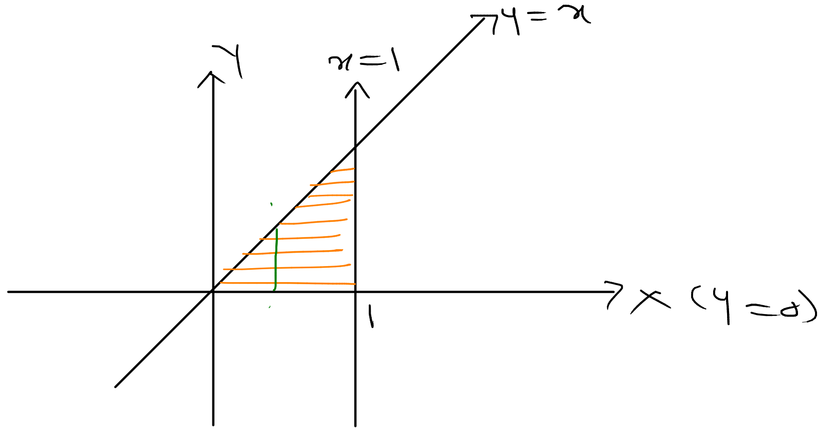
$$= -24\pi$$

Ex. Use Green's thm to find counterclockwise circulation for $\bar{F} = (x+y)\bar{i} - (x^2+y^2)\bar{j}$ where c is the triangle bdd by $y=0$, $x=1$ & $y=x$

$\Rightarrow$ circulation $= \oint_c \bar{F} . \bar{T}\, ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

$M = x+y \quad \& \quad N = -(x^2+y^2)$

$\frac{\partial M}{\partial y} = 1$ & $\frac{\partial N}{\partial x} = -2x$



$0 \leq y \leq x$ & $0 \leq x \leq 1$

$$\text{circulation} = \int_{x=0}^{1}\int_{y=0}^{x} (-2x-1)\,dy\,dx$$

$$= \int_{x=0}^{1} (-2x-1)\,[y]_{y=0}^{x}\,dx$$

$$= \int_{x=0}^{1} (-2x-1)(x)\,dx$$

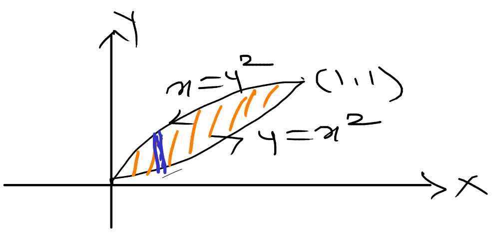
$$= \int_{x=0}^{1} (-2x^2-x)\,dx$$

$$= \left[-2\frac{x^3}{3} - \frac{x^2}{2}\right]_0^1$$

$$= \left(-\frac{2}{3} - \frac{1}{2}\right) - 0$$

$$= \frac{-4-3}{6} = -\frac{7}{6}$$

Ex. Use Green's thm to find the counterclockwise circulation for the field $\vec{F} = (xy+y^2)\vec{i} + (x-y)\vec{j}$ where C is



$x = y^2$

$\therefore y = \pm\sqrt{x}$

$\therefore y = \sqrt{x}$ (1st quadrant)

$$\Rightarrow \quad \text{circulation} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$M = xy + y^2 \;\&\; N = x - y$$

$$\text{circulation} = \iint_R [1 - (x + 2y)]\, dx\,dy$$

$$= \iint_R (1 - x - 2y)\, dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (1 - x - 2y)\, dy\,dx$$

$$= \int_{x=0}^{1} \left[y - xy - y^2 \right]_{y=x^2}^{\sqrt{x}} dx$$

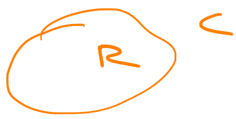
$$\int \sqrt{x} = \int x^{\frac{1}{2}} = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{x^{3/2}}{3/2}$$

$$= \int_{x=0}^{1} \left[(\sqrt{x} - x\sqrt{x} - x) - (x^2 - x^3 - x^4) \right] dx$$

$$= \left[\frac{x^{3/2}}{3/2} - \frac{x^{5/2}}{5/2} - \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} \right]_0^1$$

$$= \left(\frac{2}{3} - \frac{2}{5} - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \right) - 0$$

$$= -\frac{7}{60}$$



* Green's thm for area –

If a simple closed curve C in the plane and the region R it encloses satisfy the hypothesis of Green's thm then area of region R is given by

$$\boxed{\text{Area of } R = \frac{1}{2} \oint_C x\,dy - y\,dx}$$

Ex. Use Green's thm to find the area of the astroid $\bar{r}(t) = \cos^3 t\,\bar{i} + \sin^3 t\,\bar{j}$, $0 \le t \le 2\pi$

$\Rightarrow$ $x = \cos^3 t$ & $y = \sin^3 t$

$dx = 3\cos^2 t(-\sin t)\,dt$, $dy = 3\sin^2 t \cos t\,dt$

$$\text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx$$

$$= \frac{1}{2}\int_0^{2\pi} \cos^3 t(3\sin^2 t \cos t)\,dt - \sin^3 t(-3\cos^2 t \sin t)\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} (3\cos^4 t \sin^2 t + 3\sin^4 t \cos^2 t)\,dt$$

$$= \frac{3}{2}\int_0^{2\pi} \cos^2 t \cdot \sin^2 t\,(\underbrace{\cos^2 t + \sin^2 t}_{=1})\,dt$$

$$= \frac{3}{2}\int_0^{2\pi} \cos^2 t \cdot \sin^2 t\,dt = \frac{3}{2}\int_0 (\sin t \cos t)^2 dt$$

$$= \frac{3}{2}\int_0^{2\pi} [\tfrac{1}{2}(\underbrace{2\sin t \cos t}_{=\sin 2t})]^2\,dt$$

$$= \frac{3}{2}\int_0^{2\pi} \frac{1}{4}(\sin 2t)^2\,dt$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= \frac{3}{8}\int_0^{2\pi} \sin^2 2t\,dt$$

$$\sin^2 2t = \frac{1-\cos 4t}{2}$$

$$= \frac{3}{8}\int_0^{2\pi} \frac{1-\cos 4t}{2}\,dt$$

$$= \frac{3}{16}\left[t - \frac{\sin 4t}{4}\right]_0^{2\pi}$$

$$= \frac{3}{16}\left[2\pi - \frac{\overset{=0}{\sin 8\pi}}{4} - 0\right]$$

$$= \frac{3\pi}{8}$$

Ex. Use the Green's thm to find the area of the one arc of the cycloid

$x = t - \sin t$, $y = 1 - \cos t$

$\Rightarrow$ $x = t - \sin t$, $y = 1 - \cos t$

$dx = (1 - \cos t)\, dt$

$$\text{Area} = \oint_C y\, dx$$

$$= \int_0^{2\pi} (1 - \cos t)(1 - \cos t)\, dt$$

$$= \int_0^{2\pi} [1 - 2\cos t + \cos^2 t]\, dt$$

$$= \int_0^{2\pi} \left[1 - 2\cos t + \frac{1 + \cos 2t}{2}\right] dt$$

$$= \left[t - 2\sin t + \frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^{2\pi}$$

$$= [2\pi - 0 + \pi + 0] - 0$$

$$= 3\pi$$

Ex. Use Green's thm to find the area of circle

$\bar{r}(t) = a\cos t\, \bar{i} + a\sin t\, \bar{j}$, $0 \le t \le 2\pi$

$\Rightarrow$ $\text{Area} = \pi a^2$

Ex. Area of ellipse $\bar{r}(t) = a\cos t\, \bar{i} + b\sin t\, \bar{j}$

$0 \le t \le 2\pi$

$\Rightarrow$ $\text{Area} = \pi ab$

Refrence - A text of S.Y.B.Sc. Vector calculus by Golden series, Nirali Publication.